

A Break of the Complexity of the Numerical Approximation of Nonlinear SPDEs with Multiplicative Noise

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Abstract

A new algorithm for simulating stochastic partial differential equations (SPDEs) of evolutionary type, which is in some sense an infinite dimensional analog of Milstein's scheme for finite dimensional stochastic ordinary differential equations (SODEs), is introduced and analyzed in this article. The Milstein scheme is known to be impressively efficient for scalar one-dimensional SODEs but only for some special multidimensional SODEs due to difficult simulations of iterated stochastic integrals in the general multidimensional SODE case. It is a key observation of this article that, in contrast to what one may expect, its infinite dimensional counterpart introduced here is very easy to simulate and this, therefore, leads to a break of the complexity (number of computational operations and random variables needed to compute the scheme) in comparison to previously considered algorithms for simulating nonlinear SPDEs with multiplicative trace class noise. The analysis is supported by numerical results for a stochastic heat equation, stochastic reaction diffusion equations and a stochastic Burgers equation showing significant computational savings.

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1 Introduction

Stochastic partial differential equations (SPDEs) of evolutionary type are a fundamental instrument for modelling all kinds of dynamics with stochastic influence in nature or in man-made complex systems. Since explicit solutions of such equations are usually not available, it is a very active research topic in the last two decades to solve SPDEs approximatively. The key difficulty in this research field is the high computational complexity (in comparison to solve deterministic partial and stochastic ordinary differential equations approximatively) needed to compute an approximation of the solution of a SPDE. The high computational complexity is a consequence of the need of discretizing the continued time interval, the infinite dimensional state space of the SPDE and the infinite dimensional driving noise process. In this article a new numerical method for SPDEs with reduced computational complexity in comparison to previously considered algorithms for simulating nonlinear SPDEs with multiplicative trace class noise is introduced and analyzed. This method is in a sense an infinite dimensional analog of Milstein's scheme for finite dimensional stochastic ordinary differential equations (SODEs). The Milstein scheme is known to be impressively efficient for scalar one-dimensional SODEs but only for some special multidimensional SODEs due to difficult simulations of iterated stochastic integrals in the general multidimensional SODE case. In particular, this suggests that there is no hope to expect that an infinite dimensional analog of Milstein's scheme can be simulated efficiently and even less to expect that such a method yields a break of the computational complexity for approximating SPDEs. However, the main contribution of this article is to derive an infinite dimensional analog of Milstein's scheme that can be simulated very easily by exploiting that the infinite dimensional SPDE state space is a function space on which the noise acts multiplicatively. In contrast to Milstein's scheme for SODEs, an additional ingredient of its infinite dimensional counterpart introduced here is a mollifying exponential term approximating the dominant linear operator of the SPDE. We also mention that in the case of a linear SPDE, the splitting-up method as considered by I. Gyöngy and N. Krylov in [20] already yields a break of the computational complexity and refer to Section 4.3 for a detailed comparison of the splitting-up method and our algorithm in this article. In the rest of this introductory section we first review Milstein's scheme for SODEs, then reconsider a standard method for solving SPDEs approximatively and finally, introduce our algorithm for simulating SPDEs.

Let $T \in (0, \infty)$ be a real number, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $w = (w^1, \dots, w^m) : [0, T] \times \Omega \rightarrow \mathbb{R}^m$ with $m \in \mathbb{N} := \{1, 2, \dots\}$ be a m -dimensional standard Brownian motion with respect to a normal filtration

$(\mathcal{F}_t)_{t \in [0, T]}$. Moreover, let $d \in \mathbb{N}$, $x_0 \in \mathbb{R}^d$ and let $\mu = (\mu_1, \dots, \mu_d) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma = (\sigma_{i,j})_{\substack{i \in \{1, \dots, d\} \\ j \in \{1, \dots, m\}}} : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ be two appropriate smooth and regular functions with globally bounded derivatives (see, e.g., Theorem 10.3.5 in P. E. Kloeden and E. Platen [36] for details). The stochastic ordinary differential equation

$$dX_t = \mu(X_t) dt + \sigma(X_t) dw_t, \quad X_0 = x_0 \quad (1)$$

for all $t \in [0, T]$ then admits a unique solution. More precisely, there exists an up to indistinguishability unique adapted stochastic process $X : [0, T] \times \Omega \rightarrow \mathbb{R}^d$ with continuous sample paths which satisfies

$$\begin{aligned} X_t &= x_0 + \int_0^t \mu(X_s) ds + \int_0^t \sigma(X_s) dw_s \\ &= x_0 + \int_0^t \mu(X_s) ds + \sum_{i=1}^m \int_0^t \sigma_i(X_s) dw_s^i \quad \mathbb{P}\text{-a.s.} \end{aligned} \quad (2)$$

for all $t \in [0, T]$. Here $\sigma_i : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is given by $\sigma_i(x) = (\sigma_{1,i}(x), \dots, \sigma_{d,i}(x))$ for all $x \in \mathbb{R}^d$ and all $i \in \{1, \dots, m\}$. Milstein's method (see, e.g., (3.3) in Section 10.3 in P. E. Kloeden and E. Platen [36] and also G. N. Milstein's original article [44]) applied to the SODE (1) is then given by $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$ -measurable mappings $y_n^N : \Omega \rightarrow \mathbb{R}^d$, $n \in \{0, 1, \dots, N\}$, $N \in \mathbb{N}$, with $y_0^N = x_0$ and

$$\begin{aligned} y_{n+1}^N &= y_n^N + \frac{T}{N} \cdot \mu(y_n^N) + \sum_{i=1}^m \sigma_i(y_n^N) \cdot \left(w_{\frac{(n+1)T}{N}}^i - w_{\frac{nT}{N}}^i \right) \\ &\quad + \sum_{i,j=1}^m \sum_{k=1}^d \left(\frac{\partial}{\partial x_k} \sigma_i \right) (y_n^N) \cdot \sigma_{k,j}(y_n^N) \cdot \int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} \int_{\frac{nT}{N}}^s dw_u^j dw_s^i \quad (3) \end{aligned}$$

\mathbb{P} -a.s. for all $n \in \{0, 1, \dots, N-1\}$ and all $N \in \mathbb{N}$. Although Milstein's scheme is known to converge significantly faster than many other methods such as the Euler-Maruyama scheme, it is only of limited use due to difficult simulations of the iterated stochastic integrals $\int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} \int_{\frac{nT}{N}}^s dw_u^j dw_s^i$ for $i, j \in \{1, \dots, m\}$ with $i \neq j$, $n \in \{0, 1, \dots, N-1\}$ and $N \in \mathbb{N}$ in (3). In the special situation of so called commutative noise (see (3.13) in Section 10.3 in [36]), i.e.

$$\sum_{k=1}^d \left(\frac{\partial}{\partial x_k} \sigma_i \right) (x) \cdot \sigma_{k,j}(x) = \sum_{k=1}^d \left(\frac{\partial}{\partial x_k} \sigma_j \right) (x) \cdot \sigma_{k,i}(x) \quad (4)$$

for all $x \in \mathbb{R}^d$ and all $i, j \in \{1, \dots, m\}$, the Milstein scheme can be simplified and complicated iterated stochastic integrals in (3) can be avoided. More

precisely, in the case (4), Milstein's scheme (3) reduces to

$$\begin{aligned}
y_{n+1}^N &= y_n^N + \frac{T}{N} \cdot \mu(y_n^N) + \sum_{i=1}^m \sigma_i(y_n^N) \cdot \left(w_{\frac{(n+1)T}{N}}^i - w_{\frac{nT}{N}}^i \right) \\
&+ \frac{1}{2} \sum_{i,j=1}^m \sum_{k=1}^d \left(\frac{\partial}{\partial x_k} \sigma_i \right) (y_n^N) \cdot \sigma_{k,j}(y_n^N) \cdot \left(w_{\frac{(n+1)T}{N}}^i - w_{\frac{nT}{N}}^i \right) \cdot \left(w_{\frac{(n+1)T}{N}}^j - w_{\frac{nT}{N}}^j \right) \\
&- \frac{T}{2N} \sum_{i=1}^m \sum_{k=1}^d \left(\frac{\partial}{\partial x_k} \sigma_i \right) (y_n^N) \cdot \sigma_{k,i}(y_n^N) \quad (5)
\end{aligned}$$

\mathbb{P} -a.s. for all $n \in \{0, 1, \dots, N-1\}$ and all $N \in \mathbb{N}$ (see (3.16) in Section 10.3 in [36]). For instance, in the case $d = m = 1$, condition (4) is obviously fulfilled and the Milstein scheme (5) can then be written as

$$\begin{aligned}
y_{n+1}^N &= y_n^N + \frac{T}{N} \cdot \mu(y_n^N) + \sigma(y_n^N) \cdot \left(w_{\frac{(n+1)T}{N}} - w_{\frac{nT}{N}} \right) \\
&+ \frac{1}{2} \cdot \sigma'(y_n^N) \cdot \sigma(y_n^N) \cdot \left(\left(w_{\frac{(n+1)T}{N}} - w_{\frac{nT}{N}} \right)^2 - \frac{T}{N} \right) \quad (6)
\end{aligned}$$

\mathbb{P} -a.s. for all $n \in \{0, 1, \dots, N-1\}$ and all $N \in \mathbb{N}$ (see (3.1) in Section 10.3 in [36]). Of course, (6) can be simulated very efficiently. However, (4) is in the case of a multidimensional SODE seldom fulfilled and even if it is fulfilled, Milstein's method (5) becomes less useful if $d, m \in \mathbb{N}$ are large. For example, if $d = m = 20$ holds, then the middle term in (5) contains $20^3 = 8000$ summands. So, more than 8000 additional arithmetic operations are needed to compute y_{n+1}^N from y_n^N for $n \in \{0, 1, \dots, N-1\}$, $N \in \mathbb{N}$ via (5) in the case $d = m = 20$ in general which makes Milstein's scheme less efficient. This suggests that there is no hope to expect that an infinite dimensional analog of Milstein's method can be simulated efficiently in the case of infinite dimensional state spaces such as $L^2((0, 1), \mathbb{R})$ instead of \mathbb{R}^d and \mathbb{R}^m respectively. The purpose of this article is to demonstrate that this is not true. More precisely, an infinite dimensional analog of Milstein's scheme that can be simulated very easily is derived and analyzed here. This leads to a break of the complexity (number of computational operations and random variables needed to compute the scheme) in comparison to previously considered algorithms for simulating nonlinear SPDEs with multiplicative trace class noise which is illustrated in the following.

Let $H = L^2((0, 1), \mathbb{R})$ be the \mathbb{R} -Hilbert space of equivalence classes of $\mathcal{B}((0, 1))/\mathcal{B}(\mathbb{R})$ -measurable and Lebesgue square integrable functions from $(0, 1)$ to \mathbb{R} and let $f, b : (0, 1) \times \mathbb{R} \rightarrow \mathbb{R}$ be two appropriate smooth and

regular functions with globally bounded derivatives (see (30) and (31) for details). As usual we do not distinguish between a $\mathcal{B}((0, 1))/\mathcal{B}(\mathbb{R})$ -measurable and Lebesgue square integrable function from $(0, 1)$ to \mathbb{R} and its equivalence class in H . Moreover, let $\kappa \in (0, \infty)$ be a real number, let $\xi : [0, 1] \rightarrow \mathbb{R}$ with $\xi(0) = \xi(1) = 0$ be a smooth function and let $W : [0, T] \times \Omega \rightarrow H$ be a standard Q -Wiener process with respect to \mathcal{F}_t , $t \in [0, T]$, with a trace class operator $Q : H \rightarrow H$ (see, for instance, Definition 2.1.12 in [50]). It is a classical result (see, e.g., Proposition 2.1.5 in [50]) that the covariance operator $Q : H \rightarrow H$ of the Wiener process $W : [0, T] \times \Omega \rightarrow H$ has an orthonormal basis $g_j \in H$, $j \in \mathbb{N}$, of eigenfunctions with summable eigenvalues $\mu_j \in [0, \infty)$, $j \in \mathbb{N}$. In order to have a more concrete example, we consider the choice $g_j(x) = \sqrt{2} \sin(j\pi x)$ and $\mu_j = \frac{1}{j^2}$ for all $x \in (0, 1)$ and all $j \in \mathbb{N}$ in the following and refer to Section 2 for our general setting and to Section 4 for further possible examples. Then we consider the SPDE

$$dX_t(x) = \left[\kappa \frac{\partial^2}{\partial x^2} X_t(x) + f(x, X_t(x)) \right] dt + b(x, X_t(x)) dW_t(x) \quad (7)$$

with $X_t(0) = X_t(1) = 0$ and $X_0(x) = \xi(x)$ for $x \in (0, 1)$ and $t \in [0, T]$ on H . Under the assumptions above the SPDE (7) has a unique mild solution. Specifically, there exists an up to indistinguishability unique adapted stochastic process $X : [0, T] \times \Omega \rightarrow H$ with continuous sample path which satisfies

$$X_t = e^{At} \xi + \int_0^t e^{A(t-s)} F(X_s) ds + \int_0^t e^{A(t-s)} B(X_s) dW_s \quad \mathbb{P}\text{-a.s.} \quad (8)$$

for all $t \in [0, T]$ where $A : D(A) \subset H \rightarrow H$ is the Laplacian with Dirichlet boundary conditions times the constant $\kappa \in (0, \infty)$ and where $F : H \rightarrow H$ and $B : H \rightarrow HS(U_0, H)$ are given by $(F(v))(x) = f(x, v(x))$ and $(B(v)u)(x) = b(x, v(x)) \cdot u(x)$ for all $x \in (0, 1)$, $v \in H$ and all $u \in U_0$. Here $U_0 = Q^{\frac{1}{2}}(H)$ with $\langle v, w \rangle_{U_0} = \left\langle Q^{-\frac{1}{2}}v, Q^{-\frac{1}{2}}w \right\rangle_H$ for all $v, w \in U_0$ is the image \mathbb{R} -Hilbert space of $Q^{\frac{1}{2}}$ (see Appendix C in [50]).

Then our goal is to solve the strong approximation problem (see Section 9.3 in [36]) of the SPDE (7). More precisely, we want to compute a $\mathcal{F}/\mathcal{B}(H)$ -measurable numerical approximation $Y : \Omega \rightarrow H$ such that

$$\left(\mathbb{E} \left[\int_0^1 |X_T(x) - Y(x)|^2 dx \right] \right)^{\frac{1}{2}} < \varepsilon \quad (9)$$

holds for a given precision $\varepsilon > 0$ with the least possible computational effort (number of computational operations and independent standard normal random variables needed to compute $Y : \Omega \rightarrow H$). A computational operation is

here an arithmetic operation (addition, subtraction, multiplication, division), a trigonometric operation (sine, cosine) or an evaluation of $f : (0, 1) \times \mathbb{R} \rightarrow \mathbb{R}$, $b : (0, 1) \times \mathbb{R} \rightarrow \mathbb{R}$ or the exponential function.

In order to be able to simulate such a numerical approximation on a computer both the time interval $[0, T]$ and the infinite dimensional space $H = L^2((0, 1), \mathbb{R})$ have to be discretized. While for temporal discretizations the linear implicit Euler scheme (see [7, 8, 18, 25, 26, 27, 57, 58, 60]) and the linear implicit Crank-Nicolson scheme (see [25, 26, 57, 58]) are often used, spatial discretizations are usually achieved with finite elements (see [1, 2, 7, 8, 13, 27, 35, 38, 39, 42, 58, 60]), finite differences (see [16, 24, 43, 49, 52, 53, 54, 55, 57, 59]) and spectral Galerkin methods (see [15, 26, 29, 31, 37, 40, 41, 46, 48]). For instance, the linear implicit Euler scheme combined with spectral Galerkin methods which we denote by $\mathcal{F}/\mathcal{B}(H)$ -measurable mappings $Z_n^N : \Omega \rightarrow H$, $n \in \{0, 1, \dots, N^3\}$, $N \in \mathbb{N}$, is given by $Z_0^N = P_N(\xi)$ and

$$Z_{n+1}^N = P_N \left(I - \frac{T}{N^3} A \right)^{-1} \left(Z_n^N + \frac{T}{N^3} \cdot f(\cdot, Z_n^N) + b(\cdot, Z_n^N) \cdot \left(W_{\frac{(n+1)T}{N^3}}^N - W_{\frac{nT}{N^3}}^N \right) \right) \quad (10)$$

\mathbb{P} -a.s. for all $n \in \{0, 1, \dots, N^3 - 1\}$ and all $N \in \mathbb{N}$. Here the bounded linear operators $P_N : H \rightarrow H$, $N \in \mathbb{N}$, and the Wiener processes $W^N : [0, T] \times \Omega \rightarrow H$, $N \in \mathbb{N}$, are given by

$$(P_N(v))(x) = \sum_{n=1}^N 2 \sin(n\pi x) \int_0^1 \sin(n\pi y) v(y) dy$$

for all $x \in (0, 1)$, $v \in H$, $N \in \mathbb{N}$ and by $W_t^N(\omega) = P_N(W_t(\omega))$ for all $t \in [0, T]$, $\omega \in \Omega$, $N \in \mathbb{N}$. Moreover, we use the notations $v \cdot w : (0, 1) \rightarrow \mathbb{R}$, $v^2 : (0, 1) \rightarrow \mathbb{R}$ and $\varphi(\cdot, v) : (0, 1) \rightarrow \mathbb{R}$ given by

$$(v \cdot w)(x) = v(x) \cdot w(x), \quad (v^2)(x) = (v(x))^2, \quad (\varphi(\cdot, v))(x) = \varphi(x, v(x))$$

for all $x \in (0, 1)$ and all functions $v, w : (0, 1) \rightarrow \mathbb{R}$, $\varphi : (0, 1) \times \mathbb{R} \rightarrow \mathbb{R}$ here and below. In (10) the infinite dimensional \mathbb{R} -Hilbert space H is projected down to the N -dimensional \mathbb{R} -Hilbert space $P_N(H)$ with $N \in \mathbb{N}$ and the infinite dimensional Wiener process $W : [0, T] \times \Omega \rightarrow H$ is approximated by the finite dimensional processes $W^N : [0, T] \times \Omega \rightarrow H$, $N \in \mathbb{N}$, for the spatial discretization. For the temporal discretization in the scheme Z_n^N , $n \in \{0, 1, \dots, N^3\}$, above the time interval $[0, T]$ is divided into N^3 subintervals, i.e. N^3 time steps are used, for $N \in \mathbb{N}$. The exact solution $X : [0, T] \times \Omega \rightarrow H$ of the SPDE (7) has values in $D((-A)^\gamma)$ and satisfies

$\mathbb{E}\|(-A)^\gamma X_T\|_H^2 < \infty$ for all $\gamma \in (0, \frac{3}{4})$ (see Section 4.3 in [34]). This shows

$$\begin{aligned} (\mathbb{E}\|X_T - P_N(X_T)\|_H^2)^{\frac{1}{2}} &\leq (\mathbb{E}\|(-A)^\gamma X_T\|_H^2)^{\frac{1}{2}} \|(-A)^{-\gamma}(I - P_N)\|_{L(H)} \\ &\leq (\mathbb{E}\|(-A)^\gamma X_T\|_H^2)^{\frac{1}{2}} (1 + \kappa^{-1}) N^{-2\gamma} < \infty \end{aligned}$$

for all $N \in \mathbb{N}$ and all $\gamma \in (0, \frac{3}{4})$. So, $P_N(X_T)$ converges in the root mean square sense to X_T with order $\frac{3}{2}-$ as N goes to infinity. (For a real number $\delta \in (0, \infty)$, we write $\delta-$ for the convergence order if the convergence order is higher than $\delta - r$ for every arbitrarily small $r \in (0, \delta)$.) Additionally, the solution process of the SPDE (7) is known to be $\frac{1}{2}$ -Hölder continuous in the root mean square sense (see, for instance, Theorem 1 in [34]) and therefore, the linear implicit Euler scheme converges temporally in the root mean square to the exact solution of the SPDE (7) with order $\frac{1}{2}$ (see, e.g., Theorem 1.1 in [60]). Combining the convergence rate $\frac{3}{2}-$ for the spatial discretization and the convergence rate $\frac{1}{2}$ for the temporal discretization indicates that it is asymptotically optimal to use the cubic number N^3 of time steps in the linear implicit Euler scheme Z_n^N , $n \in \{0, 1, \dots, N^3\}$, above.

We now review how efficiently the numerical method (10) solves the strong approximation problem (9) of the SPDE (7). Standard results in the literature (see, for instance, Theorem 2.1 in [26]) yield the existence of real numbers $C_r > 0$, $r \in (0, \frac{3}{2})$, such that

$$\left(\mathbb{E} \left[\int_0^1 |X_T(x) - Z_{N^3}^N(x)|^2 dx \right] \right)^{\frac{1}{2}} \leq C_r \cdot N^{(r-\frac{3}{2})} \quad (11)$$

holds for all $N \in \mathbb{N}$ and all arbitrarily small $r \in (0, \frac{3}{2})$. The linear implicit Euler approximation $Z_{N^3}^N$ thus converges in the root mean square sense to X_T with order $\frac{3}{2}-$ as N goes to infinity. Moreover, since $P_N(H)$ is N -dimensional and since N^3 time steps are used in (10), $O(N^4 \log(N))$ computational operations and random variables are needed to compute $Z_{N^3}^N$. The logarithmic term in $O(N^4 \log(N))$ arises due to computing the nonlinearities f and b with fast Fourier transform where aliasing errors are neglected here and below. Combining the computational effort $O(N^4 \log(N))$ and the convergence order $\frac{3}{2}-$ in (11) shows that the linear implicit Euler scheme needs about $\mathbf{O}(\varepsilon^{-\frac{8}{3}})$ computational operations and independent standard normal random variables to achieve the desired precision $\varepsilon > 0$ in (9). In fact, we have demonstrated that Euler's method (10) needs $O(\varepsilon^{-(\frac{8}{3}+r)})$ computational operations and random variables to solve (9) for every arbitrarily small $r \in (0, \infty)$ but for simplicity we write about $O(\varepsilon^{-\frac{8}{3}})$ computational operations and random variables here and below.

Having reviewed Euler's method (10), we now derive our infinite dimensional analog of Milstein's scheme. In the finite dimensional SODE case, the Milstein scheme (3) is derived by applying Itô's formula to the integrand process $\sigma(X_t)$, $t \in [0, T]$, in (2). This approach is based on the fact that the diffusion coefficient σ is a smooth test function and that the solution process of (1) is a Itô process. This strategy is not directly available in infinite dimensions since (7) does in general not admit a strong solution to which Itô's formula could be applied. Recently, in [32] in the case of additive noise and in [30] in the general case, this problem has been overcome by first applying Taylor's formula in Banach spaces to the diffusion coefficient B in the mild integral equation (8) and by then inserting a lower order approximation recursively (see Section 4.3 in [30]). More formally, using $F(X_s) \approx F(X_0)$ and $B(X_s) \approx B(X_0) + B'(X_0)(X_s - X_0)$ for $s \in [0, T]$ in (8) shows

$$\begin{aligned} X_t &\approx e^{At}\xi + \int_0^t e^{A(t-s)}F(X_0)ds + \int_0^t e^{A(t-s)}B(X_0)dW_s \\ &\quad + \int_0^t e^{A(t-s)}B'(X_0)(X_s - X_0)dW_s \\ &\approx e^{At}\left(X_0 + t \cdot F(X_0) + \int_0^t B(X_0)dW_s + \int_0^t B'(X_0)(X_s - X_0)dW_s\right) \end{aligned}$$

for $t \in [0, T]$. The estimate $X_s \approx X_0 + \int_0^s B(X_0)dW_u$ for $s \in [0, T]$ then gives

$$X_t \approx e^{At}\left(X_0 + t \cdot F(X_0) + \int_0^t B(X_0)dW_s + \int_0^t B'(X_0)\left(\int_0^s B(X_0)dW_u\right)dW_s\right) \quad (12)$$

for $t \in [0, T]$. Using Itô's formula this temporal approximation has already been obtained in (1.12) in [45] under additional smoothness assumptions of the driving noise process of the SPDE (8) (see Assumption C in [45]) which guarantee the existence of a strong solution and thus allow the application of Itô's formula. Combining the temporal approximation (12) and the spatial discretization in (10) indicates the numerical scheme with $\mathcal{F}/\mathcal{B}(H)$ -measurable mappings $Y_n^N : \Omega \rightarrow H$, $n \in \{0, 1, \dots, N^2\}$, $N \in \mathbb{N}$, given by $Y_0^N = P_N(\xi)$ and

$$\begin{aligned} Y_{n+1}^N &= P_N e^{A \frac{T}{N^2}} \left(Y_n^N + \frac{T}{N^2} \cdot F(Y_n^N) + B(Y_n^N) \left(W_{\frac{(n+1)T}{N^2}}^N - W_{\frac{nT}{N^2}}^N \right) \right. \\ &\quad \left. + \int_{\frac{nT}{N^2}}^{\frac{(n+1)T}{N^2}} B'(Y_n^N) \left(\int_{\frac{nT}{N^2}}^s B(Y_n^N) dW_u^N \right) dW_s^N \right) \quad (13) \end{aligned}$$

\mathbb{P} -a.s. for all $n \in \{0, 1, \dots, N^2 - 1\}$ and all $N \in \mathbb{N}$. Now, we are at a stage similar to the finite dimensional case (3): a higher order method seems to be derived which nevertheless seems to be of limited use due to the iterated high dimensional stochastic integral in (13). However, a key observation of this article is the following formula

$$\begin{aligned} & \int_{\frac{nT}{N^2}}^{\frac{(n+1)T}{N^2}} B'(Y_n^N) \left(\int_{\frac{nT}{N^2}}^s B(Y_n^N) dW_u^N \right) dW_s^N \\ &= \frac{1}{2} \left(\frac{\partial}{\partial y} b \right) (\cdot, Y_n^N) \cdot b(\cdot, Y_n^N) \cdot \left(\left(W_{\frac{(n+1)T}{N^2}}^N - W_{\frac{nT}{N^2}}^N \right)^2 - \frac{T}{N^2} \sum_{i=1}^N \mu_i(g_i)^2 \right) \end{aligned} \quad (14)$$

\mathbb{P} -a.s. for all $n \in \{0, 1, \dots, N^2 - 1\}$ and all $N \in \mathbb{N}$ (see Subsection 6.7 for the proof of the iterated integral identity (14) and see below for a heuristic explanation of this fact). So, the iterated high dimensional stochastic integral in (13) reduces to a simple product of functions. The function $\frac{\partial}{\partial y} b : (0, 1) \times \mathbb{R} \rightarrow \mathbb{R}$ is here the partial derivative $(\frac{\partial}{\partial y} b)(x, y)$ for $x \in (0, 1)$ and $y \in \mathbb{R}$. Using (14) the numerical scheme (13) thus reduces to

$$\begin{aligned} Y_{n+1}^N &= P_N e^{A \frac{T}{N^2}} \left(Y_n^N + \frac{T}{N^2} \cdot f(\cdot, Y_n^N) + b(\cdot, Y_n^N) \cdot \left(W_{\frac{(n+1)T}{N^2}}^N - W_{\frac{nT}{N^2}}^N \right) \right. \\ &\quad \left. + \frac{1}{2} \left(\frac{\partial}{\partial y} b \right) (\cdot, Y_n^N) \cdot b(\cdot, Y_n^N) \cdot \left(\left(W_{\frac{(n+1)T}{N^2}}^N - W_{\frac{nT}{N^2}}^N \right)^2 - \frac{T}{N^2} \sum_{i=1}^N \mu_i(g_i)^2 \right) \right) \end{aligned} \quad (15)$$

\mathbb{P} -a.s. for all $n \in \{0, 1, \dots, N^2 - 1\}$ and all $N \in \mathbb{N}$. It can be seen that only increments of the finite dimensional Wiener processes $W^N : [0, T] \times \Omega \rightarrow H$, $N \in \mathbb{N}$, are used in (15). Note that, as in the case of (10), the infinite dimensional \mathbb{R} -Hilbert space H is projected down to the N -dimensional \mathbb{R} -Hilbert space $P_N(H)$ with $N \in \mathbb{N}$ and the infinite dimensional Wiener process $W : [0, T] \times \Omega \rightarrow H$ is approximated by the finite dimensional Wiener processes $W^N : [0, T] \times \Omega \rightarrow H$, $N \in \mathbb{N}$, for the spatial discretization in (15). For the temporal discretization in the scheme Y_n^N , $n \in \{0, 1, \dots, N^2\}$, above the time interval $[0, T]$ is divided into N^2 subintervals, i.e. N^2 instead of N^3 time steps are used in (15), for $N \in \mathbb{N}$. In the following we explain why it is crucial to use N^2 time steps in (15) instead of N^3 time steps in the case of the linear implicit Euler scheme (10).

More formally, we now illustrate how efficiently the method (15) solves the strong approximation problem (9) of the SPDE (7). Theorem 1 (see

Section 3 below) gives the existence of real numbers $C_r > 0$, $r \in (0, \frac{3}{2})$, such that

$$\left(\mathbb{E} \left[\int_0^1 |X_T(x) - Y_{N^2}^N(x)|^2 dx \right] \right)^{\frac{1}{2}} \leq C_r \cdot N^{(r-\frac{3}{2})} \quad (16)$$

holds for all $N \in \mathbb{N}$ and all arbitrarily small $r \in (0, \frac{3}{2})$. The approximation $Y_{N^2}^N$ thus converges in the root mean square sense to X_T with order $\frac{3}{2}-$ as N goes to infinity. The expression

$$\frac{1}{2} \left(\frac{\partial}{\partial y} b \right) (\cdot, Y_n^N) \cdot b(\cdot, Y_n^N) \cdot \left(\left(W_{\frac{(n+1)T}{N}}^N - W_{\frac{nT}{N}}^N \right)^2 - \frac{T}{N^2} \sum_{i=1}^N \mu_i(g_i)^2 \right) \quad (17)$$

for $n \in \{0, 1, \dots, N^2 - 1\}$ and $N \in \mathbb{N}$ in (15) contains additional information of the solution process of (7) and this allows us to use less time steps, N^2 in (15) instead of N^3 in (10), to achieve the same convergence rate as the linear implicit Euler scheme (10) (compare (11) and (16)). Nonetheless, (17) and hence the numerical method (15) can be simulated very easily. The function $\frac{T}{N^2} \sum_{i=1}^N \mu_i(g_i)^2$ in (17) can be computed once in advance for which $O(N^2)$ computational operations are needed. Having computed $\frac{T}{N^2} \sum_{i=1}^N \mu_i(g_i)^2$, $O(N \log(N))$ further computational operations and random variables are needed to compute (17) from Y_n^N for one fixed $n \in \{0, 1, \dots, N^2 - 1\}$ by using fast Fourier transform. Since $O(N \log(N))$ computational operations and random variables are needed for one time step and since N^2 time steps are used in (15), $O(N^3 \log(N))$ computational operations and random variables are needed to compute $Y_{N^2}^N$. Combining the computational effort $O(N^3 \log(N))$ and the convergence order $\frac{3}{2}-$ in (16) shows that the numerical method (15) needs about $\mathbf{O}(\varepsilon^{-2})$ computational operations and independent standard normal random variables to achieve the desired precision $\varepsilon > 0$ in (9). To sum up, the algorithm (15) reduces the complexity of the problem (9) of the SPDE (7) from about $\mathbf{O}(\varepsilon^{-\frac{8}{3}})$ to about $\mathbf{O}(\varepsilon^{-2})$.

However, the complexity rates $O(\varepsilon^{-\frac{8}{3}})$ and $O(\varepsilon^{-2})$ are both asymptotic results as $\varepsilon > 0$ tends to zero. Therefore, from a practical point of view, one may ask whether the algorithm (15) solves the strong approximation problem (9) more efficiently than the linear implicit Euler scheme (10) for a given concrete $\varepsilon > 0$ and a given example of the form (7). In order to analyze this question we compare both methods in the case of a simple stochastic reaction diffusion equation. More formally, let $\kappa = \frac{1}{100}$, let $\xi : [0, 1] \rightarrow \mathbb{R}$ be given by $\xi(x) = 0$ for all $x \in [0, 1]$ and suppose that $f, b : (0, 1) \times \mathbb{R} \rightarrow \mathbb{R}$ are given by $f(x, y) = 1 - y$ and $b(x, y) = \frac{1-y}{1+y^2}$ for all $x \in (0, 1)$, $y \in \mathbb{R}$. The

SPDE (7) thus reduces to

$$dX_t(x) = \left[\frac{1}{100} \frac{\partial^2}{\partial x^2} X_t(x) + 1 - X_t(x) \right] dt + \frac{1 - X_t(x)}{1 + X_t(x)^2} dW_t(x) \quad (18)$$

with $X_t(0) = X_t(1) = 0$ and $X_0 = 0$ for $x \in (0, 1)$ and $t \in [0, 1]$ (see also Section 4.1 for more details concerning this example). Additionally, assume that (9) for the SPDE (18) should be solved with the precision of say three decimals, i.e. with the precision $\varepsilon = \frac{1}{1000}$ in (9). In Figure 1 the approximation error in the sense of (9) of the linear implicit Euler approximation $Z_{N^3}^N$ (see (10)) and of the approximation $Y_{N^2}^N$ (see (15)) is plotted against the precise number of independent standard normal random variables needed to compute the corresponding approximation for $N \in \{2, 4, 8, 16, 32, 64, 128\}$: It turns out that $Z_{128^3}^{128}$ in the case of the linear implicit Euler scheme (10) and that $Y_{128^2}^{128}$ in the case of the algorithm (15) achieve the desired precision $\varepsilon = \frac{1}{1000}$ in (9) for the SPDE (18). The MATLAB codes for simulating $Z_{128^3}^{128}$ via (10) and $Y_{128^2}^{128}$ via (15) for the SPDE (18) are presented below in Figure 2 and Figure 3 respectively. The differences of the codes and the additional code needed for the new algorithm (15) are printed bold in Figure 3. The MATLAB code in Figure 2 requires on an INTEL PENTIUM D a CPU time of about **15 minutes and 25.03 seconds** (925.03 seconds) while the code in Figure 3 requires a CPU time of about **8.93 seconds** to be evaluated on the same computer. So, the algorithm (15) is for the SPDE (18) more than **hundred times faster** than the linear implicit Euler scheme (10) in order to achieve a precision of three decimals in (9). To sum up, **the four code changes (line 1, lines 3-6, line 10 and line 11) in Figure 3 enables us to simulate (18) with the precision of three decimals in about 9 seconds instead of about 15 minutes.** Further numerical examples for the algorithm (15) can be found in Section 4 and Section 5.

Having illustrated the efficiency of the method (15), we now take a short look into the literature of numerical analysis for SPDEs. First, it should be mentioned that any combination of finite elements, finite differences or spectral Galerkin methods for the spatial discretization and the linear implicit Euler scheme or also the linear implicit Crank-Nicolson scheme for the temporal discretization do not reduce the complexity $O(\varepsilon^{-\frac{8}{3}})$ of the problem (9) of the SPDE (7). However, in the case of a linear SPDE, the splitting-up method as considered by I. Gyöngy and N. Krylov in [20] (see also [3, 4, 12, 19, 21, 22, 23] and the references therein) converges with a higher temporal order and therefore breaks the computational complexity in comparison to the linear implicit Euler scheme. The key idea of the splitting-up method is to split the considered SPDE into appropriate subequations that are easier to solve than the original SPDE, e.g., that can be solved explicitly. The splitting-up method

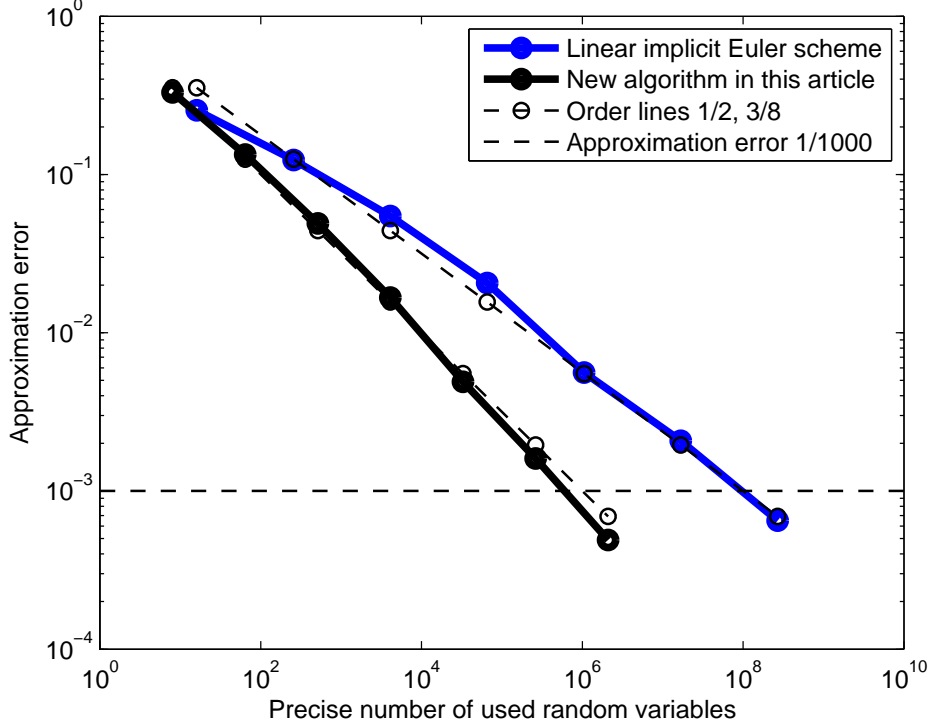


Figure 1: SPDE (18): Approximation error in the sense of (9) of the linear implicit Euler approximation $Z_{N^3}^N$ (see (10)) and of the approximation $Y_{N^2}^N$ (see (15)) against the precise number of independent standard normal random variables needed to compute the corresponding approximation for $N \in \{2, 4, 8, 16, 32, 64, 128\}$.

thus essentially depends on the simplicity of the splitted subequations and can therefore in general not be used efficiently for nonlinear SPDEs which are investigated in this article. Nonetheless, in the case of a linear SPDE as in I. Gyöngy and N. Krylov's article [20], the splitting-up method and the scheme in this article converge with the same complexity rate (see Section 4.3 for a more detailed comparison of the splitting-up method and the algorithm in this article). Additionally, in the case $f = 0$ in (7), T. Müller-Gronbach and K. Ritter invented a new scheme which reduces the number of random variables needed for solving a similar problem as (9) from about $O(\varepsilon^{-\frac{8}{3}})$ to about $O(\varepsilon^{-2})$ (see [46] and also [47]). Nonetheless, the number of computational operations needed and thus the overall computational complexity could not be reduced by their algorithm. Moreover, Milstein type schemes for SPDEs have been considered in [5, 15, 37, 45]. In [15], W. Grecksch

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1 N = 128; M = N^3; A = -pi^2*(1:N).^2/100; Y = zeros(1,N);
2 mu = (1:N).^2; f = @(x) 1-x; b = @(x) (1-x)./(1+x.^2);
3 for m=1:M
4     y = dst(Y) * sqrt(2);
5     dW = dst( randn(1,N) .* sqrt(mu*2/M) );
6     y = y + f(y)/M + b(y).*dW;
7     Y = idst( y ) / sqrt(2) ./ ( 1 - A/M );
8 end
9 plot( (0:N+1)/(N+1), [0,dst(Y)*sqrt(2),0] );

```

Figure 2: MATLAB code for simulating the linear implicit Euler approximation $Z_{N^3}^N$ with $N = 128$ (see (10)) for the SPDE (18).

and P. E. Kloeden proposed a Milstein like scheme for a SPDE driven by a scalar one-dimensional Brownian motion (see also [37]). In view of (6), their Milstein scheme can be simulated efficiently since the driving noise process is one-dimensional. Furthermore, in the case of a linear SPDE, P.-L. Chow et al. constructed in the interesting article [5] a scheme similar to (13) but with an additional term. (The additional term may be useful for decreasing the error constant but turns out not to be needed in order to achieve the higher approximation order due to Theorem 1 here.) In order to simulate the iterated stochastic integral in their scheme, they then suggest to omit the summands in the double sum in (5) for which $i \neq j$ holds (see (2.8) in [5]). Their idea thus yields a scheme that can be simulated very efficiently but does in general not converge with a higher order anymore except for a linear SPDE driven by a scalar one-dimensional Brownian motion. Finally, based on Itô's formula, Y. S. Mishura and G. M. Shevchenko proposed in [45] the temporal approximation (12) under additional smoothness assumptions of the driving noise process of the SPDE (8) which guarantee the existence of a strong solution and thus allow the application of Itô's formula (see Assumption C in [45]). The simulation of the iterated stochastic integrals in their numerical approximation remained an open question (see Remark 1.1 in [45]). To sum up, to the best of our knowledge the computational complexity barrier $O(\varepsilon^{-\frac{8}{3}})$ for the problem (9) of the SPDE (7) has not been broken by any algorithm proposed in the literature yet.

It is in some sense amazing that the Milstein type scheme (13) reduces to the simple algorithm (15) in the infinite dimensional SPDE setting (7) which is the key observation in order to beat the computational complexity barrier $O(\varepsilon^{-\frac{8}{3}})$. The reason for this simplification is that $B(v) \in HS(U_0, H)$ for $v \in H$ does not act as an arbitrary linear operator on the infinite di-

```

1 N = 128; M = N^2; A = -pi^2*(1:N).^2/100; Y = zeros(1,N);
2 mu = (1:N).^ -2; f = @(x) 1-x; b = @(x) (1-x)./(1+x.^2);
3 bb = @(x) (1-x).*(x.^2-2*x-1)/2./(1+x.^2).^3; g = zeros(1,N);
4 for n=1:N
5     g = g+2*sin(n*(1:N)/(N+1)*pi).^2*mu(n)/M;
6 end
7 for m=1:M
8     y = dst(Y) * sqrt(2);
9     dW = dst( randn(1,N) .* sqrt(mu*2/M) );
10    y = y + f(y)/M + b(y).*dW + bb(y).*(dW.^2 - g);
11    Y = exp( A/M ) .* idst( y ) / sqrt(2);
12 end
13 plot( (0:N+1)/(N+1), [0,dst(Y)*sqrt(2),0] );

```

Figure 3: MATLAB code for simulating the approximation $Y_{N^2}^N$ with $N = 128$ (see (15)) for the SPDE (18).

mensional vector space U_0 but as a multiplication operator on the function space $U_0 \subset L^2((0,1), \mathbb{R})$. First, this ensures that $B : H \rightarrow HS(U_0, H)$ naturally falls into the infinite dimensional analog of the commutative noise case (4) although $b : (0,1) \times \mathbb{R} \rightarrow \mathbb{R}$ and hence $B : H \rightarrow HS(U_0, H)$ are possibly nonlinear mappings (see (34) for details). Second, in the infinite dimensional commutative noise setting, the acting of $B(v)$ for $v \in H$ as a multiplication operator assures that the infinite dimensional analog of (5) even simplifies in infinite dimensions to (15). These two facts guarantee that the method (13) reduces to the simple algorithm (15) which is in a sense the main contribution of this article. To sum up, due to the reduced complexity, the algorithm (15) provides an in comparison to previously considered numerical methods impressive instrument for simulating nonlinear stochastic partial differential equations with multiplicative trace class noise.

The rest of this article is organized as follows. In Section 2 the setting and the assumptions used are formulated. The numerical method and its convergence result (Theorem 1) are presented in Section 3. In Section 4 several examples of Theorem 1 including a stochastic heat equation and stochastic reaction diffusion equations are considered. Although our setting in Section 2 uses the standard global Lipschitz assumptions on the nonlinear coefficients of the SPDE, we demonstrate the efficiency of our method numerically for a stochastic Burgers equation with a non-globally Lipschitz nonlinearity in Section 5. The proof of Theorem 1 is postponed to Section 6.

Finally, we would like to add some concluding remarks. There are a num-

ber of directions for further research arising from this work. One is to analyze whether the exponential term in (15) can be replaced by a simpler mollifier such as $(I - \frac{T}{N^2}A)^{-1}$ for $N \in \mathbb{N}$. This would make the scheme even simpler to simulate. A second direction is to combine the temporal approximation in (15) with other spatial discretizations such as finite elements. This makes it possible to handle more complicated multidimensional domains on which the eigenfunctions of the Laplacian are not known explicitly. A third direction is to reduce the Lipschitz assumptions in Section 2 in order to handle the stochastic Burgers equation in Section 5 and further SPDEs with non-globally Lipschitz nonlinearities such as the stochastic porous medium equation and hyperbolic SPDEs. Finally, a combination of Giles' multilevel Monte Carlo approach in [14] with the method in this article should yield a break of the computational complexity in comparison to previously considered algorithms for solving the weak approximation problem (see, e.g., Section 9.4 in [36]) of the SPDE (7).

2 Setting and assumptions

Throughout this article suppose that the following setting and the following assumptions are fulfilled. Fix $T \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0, T]}$ and let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ and $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$ be two separable \mathbb{R} -Hilbert spaces. Moreover, let $Q : U \rightarrow U$ be a trace class operator and let $W : [0, T] \times \Omega \rightarrow U$ be a standard Q -Wiener process with respect to $(\mathcal{F}_t)_{t \in [0, T]}$.

Assumption 1 (Linear operator A). *Let \mathcal{I} be a finite or countable set and let $(\lambda_i)_{i \in \mathcal{I}} \subset (0, \infty)$ be a family of real numbers with $\inf_{i \in \mathcal{I}} \lambda_i \in (0, \infty)$. Moreover, let $(e_i)_{i \in \mathcal{I}}$ be an orthonormal basis of H and let $A : D(A) \subset H \rightarrow H$ be a linear operator with*

$$Av = \sum_{i \in \mathcal{I}} -\lambda_i \langle e_i, v \rangle_H e_i \quad (19)$$

for every $v \in D(A)$ and with $D(A) = \{w \in H \mid \sum_{i \in \mathcal{I}} |\lambda_i|^2 |\langle e_i, w \rangle_H|^2 < \infty\}$.

By $V_r := D((-A)^r)$ equipped with the norm $\|v\|_{V_r} := \|(-A)^r v\|_H$ for all $v \in V_r$ and all $r \in [0, \infty)$ we denote the \mathbb{R} -Hilbert spaces of domains of fractional powers of the linear operator $-A : D(A) \subset H \rightarrow H$.

Assumption 2 (Drift term F). *Let $\beta \in [0, 1)$ be a real number and let $F : V_\beta \rightarrow H$ be a twice continuously Fréchet differentiable mapping with $\sup_{v \in V_\beta} \|F'(v)\|_{L(H)} < \infty$ and $\sup_{v \in V_\beta} \|F''(v)\|_{L^{(2)}(V_\beta, H)} < \infty$.*

In order to formulate the assumption on the diffusion coefficient of our SPDE, we denote by $(U_0, \langle \cdot, \cdot \rangle_{U_0}, \|\cdot\|_{U_0})$ the separable \mathbb{R} -Hilbert space $U_0 := Q^{\frac{1}{2}}(U)$ with $\langle v, w \rangle_{U_0} = \left\langle Q^{-\frac{1}{2}}v, Q^{-\frac{1}{2}}w \right\rangle_U$ for all $v, w \in U_0$ (see, for example, Section 2.3.2 in [50]). For an arbitrary bounded linear operator $S \in L(U)$, we denote by $S^{-1} : \text{im}(S) \subset U \rightarrow U$ the pseudo inverse of S (see, for instance, Appendix C in [50]).

Assumption 3 (Diffusion term B). *Let $B : V_\beta \rightarrow HS(U_0, H)$ be a twice continuously Fréchet differentiable mapping with $\sup_{v \in V_\beta} \|B'(v)\|_{L(H, HS(U_0, H))} < \infty$ and $\sup_{v \in V_\beta} \|B''(v)\|_{L^{(2)}(V_\beta, HS(U_0, H))} < \infty$. Moreover, let $\alpha, c \in (0, \infty)$, $\delta, \vartheta \in (0, \frac{1}{2})$, $\gamma \in [\max(\delta, \beta), \delta + \frac{1}{2})$ be real numbers, let $B(V_\delta) \subset HS(U_0, V_\delta)$ and suppose that*

$$\|B(u)\|_{HS(U_0, V_\delta)} \leq c(1 + \|u\|_{V_\delta}), \quad (20)$$

$$\|B'(v)B(v) - B'(w)B(w)\|_{HS^{(2)}(U_0, H)} \leq c\|v - w\|_H, \quad (21)$$

$$\left\| (-A)^{-\vartheta} B(v) Q^{-\alpha} \right\|_{HS(U_0, H)} \leq c(1 + \|v\|_{V_\gamma}) \quad (22)$$

holds for every $u \in V_\delta$ and every $v, w \in V_\gamma$. Finally, let the bilinear Hilbert-Schmidt operator $B'(v)B(v) \in HS^{(2)}(U_0, H)$ be symmetric for every $v \in V_\beta$.

The operator $B'(v)B(v) : U_0 \times U_0 \rightarrow H$ given by

$$(B'(v)B(v))(u, \tilde{u}) = \{B'(v)(B(v)u)\}(\tilde{u})$$

for all $u, \tilde{u} \in U_0$ is a bilinear Hilbert-Schmidt operator in $HS^{(2)}(U_0, H) \cong HS(\overline{U_0 \otimes U_0}, H)$ for every $v \in V_\beta$. Therefore, condition (21) means that the mapping

$$V_\gamma \rightarrow HS^{(2)}(U_0, H), \quad v \mapsto B'(v)B(v), \quad v \in V_\gamma,$$

is globally Lipschitz continuous with respect to $\|\cdot\|_H$ and $\|\cdot\|_{HS^{(2)}(U_0, H)}$. We also would like to point out that the assumed symmetry of $B'(v)B(v) \in HS^{(2)}(U_0, H)$ for all $v \in V_\beta$ is the abstract (infinite dimensional) analog of (4). More formally, if $H = \mathbb{R}^d$, $U = \mathbb{R}^m$ and $Q = I$ with $d, m \in \mathbb{N}$ holds, then the symmetry of $B'(v)B(v) \in HS^{(2)}(\mathbb{R}^m, \mathbb{R}^d)$ reduces to (4) (with σ replaced by B). Although (4) is seldom fulfilled for finite dimensional SODEs, the symmetry of $B'(v)B(v) \in HS^{(2)}(U_0, H)$ for all $v \in V_\beta$ is naturally met in the case of multiplicative noise on infinite dimensional function spaces as we will demonstrate in Sections 4 and 5. Finally, we emphasize that we assumed in no way that the linear operators $A : D(A) \subset H \rightarrow H$ and $Q : U \rightarrow U$ are simultaneously diagonalizable.

Assumption 4 (Initial value ξ). *Let $\xi : \Omega \rightarrow V_\gamma$ be a $\mathcal{F}_0/\mathcal{B}(V_\gamma)$ -measurable mapping with $\mathbb{E} \|\xi\|_{V_\gamma}^4 < \infty$.*

These assumptions suffice to ensure the existence of a unique solution of the SPDE (23).

Proposition 1 (Existence of the solution). *Let Assumptions 1-4 in Section 2 be fulfilled. Then there exists an up to modifications unique predictable stochastic process $X : [0, T] \times \Omega \rightarrow V_\gamma$ which fulfills $\sup_{t \in [0, T]} \mathbb{E} \|X_t\|_{V_\gamma}^4 < \infty$, $\sup_{t \in [0, T]} \mathbb{E} \|B(X_t)\|_{HS(U_0, V_\delta)}^4 < \infty$ and*

$$X_t = e^{At}\xi + \int_0^t e^{A(t-s)}F(X_s)ds + \int_0^t e^{A(t-s)}B(X_s)dW_s \quad \mathbb{P}\text{-a.s.} \quad (23)$$

for all $t \in [0, T]$. Moreover, we have

$$\sup_{\substack{t_1, t_2 \in [0, T] \\ t_1 \neq t_2}} \frac{(\mathbb{E} \|X_{t_2} - X_{t_1}\|_{V_r}^4)^{\frac{1}{4}}}{|t_2 - t_1|^{\min(\gamma-r, \frac{1}{2})}} < \infty \quad (24)$$

for every $r \in [0, \gamma]$.

Proposition 1 immediately follows from Theorem 1 in [34].

3 Numerical scheme and main result

In this section our numerical method is introduced and its convergence result is stated. To this end let \mathcal{J} be a finite or countable set, let $(g_j)_{j \in \mathcal{J}} \subset U$ be an orthonormal basis of eigenfunctions of $Q : U \rightarrow U$ and let $(\mu_j)_{j \in \mathcal{J}} \subset [0, \infty)$ be the corresponding family of eigenvalues (such an orthonormal basis of eigenfunctions exists since $Q : U \rightarrow U$ is a trace class operator, see Proposition 2.1.5 in [50]). In particular, we have

$$Qu = \sum_{j \in \mathcal{J}} \mu_j \langle g_j, u \rangle_U g_j \quad (25)$$

for all $u \in U$. Additionally, let $(\mathcal{I}_N)_{N \in \mathbb{N}}$ and $(\mathcal{J}_K)_{K \in \mathbb{N}}$ be sequences of finite subsets of \mathcal{I} and \mathcal{J} respectively. Then we define the linear projection operators $P_N : H \rightarrow H$, $N \in \mathbb{N}$, by $P_N(v) := \sum_{i \in \mathcal{I}_N} \langle e_i, v \rangle_H e_i$ for all $v \in H$ and all $N \in \mathbb{N}$. Furthermore, we define Wiener processes $W^K : [0, T] \times \Omega \rightarrow U_0$, $K \in \mathbb{N}$, by

$$W_t^K(\omega) := \sum_{\substack{j \in \mathcal{J}_K \\ \mu_j \neq 0}} \langle g_j, W_t(\omega) \rangle_U g_j \quad (26)$$

for all $t \in [0, T]$, $\omega \in \Omega$ and all $K \in \mathbb{N}$. We also use the $\mathcal{F}/\mathcal{B}(U_0)$ -measurable mappings $\Delta W_m^{M,K} : \Omega \rightarrow U_0$, $m \in \{0, 1, \dots, M-1\}$, $M, K \in \mathbb{N}$, given by $\Delta W_m^{M,K}(\omega) := W_{\frac{(m+1)T}{M}}^K(\omega) - W_{\frac{mT}{M}}^K(\omega)$ for all $\omega \in \Omega$, $m \in \{0, 1, \dots, M-1\}$ and all $M, K \in \mathbb{N}$. Our numerical scheme which we denote by $\mathcal{F}/\mathcal{B}(H)$ -measurable mappings $Y_m^{N,M,K} : \Omega \rightarrow H_N$, $m \in \{0, 1, \dots, M\}$, $N, M, K \in \mathbb{N}$, is then given by $Y_0^{N,M,K} := P_N(\xi)$ and

$$\begin{aligned} Y_{m+1}^{N,M,K} := & P_N e^{A \frac{T}{M}} \left(Y_m^{N,M,K} + \frac{T}{M} \cdot F(Y_m^{N,M,K}) + B(Y_m^{N,M,K}) \Delta W_m^{M,K} \right. \\ & + \frac{1}{2} B'(Y_m^{N,M,K}) \left(B(Y_m^{N,M,K}) \Delta W_m^{M,K} \right) \Delta W_m^{M,K} \\ & \left. - \frac{T}{2M} \sum_{\substack{j \in \mathcal{J}_K \\ \mu_j \neq 0}} \mu_j B'(Y_m^{N,M,K}) \left(B(Y_m^{N,M,K}) g_j \right) g_j \right) \quad (27) \end{aligned}$$

for every $m \in \{0, 1, \dots, M-1\}$ and every $N, M, K \in \mathbb{N}$. It can be seen that only increments of the Wiener processes $W^K : [0, T] \times \Omega \rightarrow U_0$, $K \in \mathbb{N}$, are used in the scheme above and we emphasize that for many SPDEs the method (27) is very easy to simulate and implement (see Sections 1, 4 and 5 for several examples). We now present the convergence result of the scheme (27).

Theorem 1 (Main result). *Let Assumptions 1-4 in Section 2 be fulfilled. Then there is a real number $C \in (0, \infty)$ such that*

$$\begin{aligned} & \left(\mathbb{E} \left\| X_{\frac{mT}{M}} - Y_m^{N,M,K} \right\|_H^2 \right)^{\frac{1}{2}} \\ & \leq C \left(\left(\inf_{i \in \mathcal{I} \setminus \mathcal{I}_N} \lambda_i \right)^{-\gamma} + \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^\alpha + M^{-\min(2(\gamma-\beta), \gamma)} \right) \quad (28) \end{aligned}$$

holds for every $m \in \{0, 1, \dots, M\}$ and every $N, M, K \in \mathbb{N}$.

We now explain the result of Theorem 1 more detailed. The root mean square difference $(\mathbb{E} \|X_{\frac{mT}{M}} - Y_m^{N,M,K}\|_H^2)^{\frac{1}{2}}$ for $m \in \{0, 1, \dots, M\}$ and for $N, M, K \in \mathbb{N}$ of the exact solution of the SPDE (23) and of the numerical solution (27) is estimated in (28) by a constant times the sum of three terms. The first term, i.e. $(\inf_{i \in \mathcal{I} \setminus \mathcal{I}_N} \lambda_i)^{-\gamma}$ for $N \in \mathbb{N}$, arises due to discretizing the exact solution spatially, i.e. due to $\mathbb{E} \|X_t - P_N(X_t)\|_H^2$ for $N \in \mathbb{N}$ and $t \in [0, T]$. The second expression, i.e. $(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j)^\alpha$ for $K \in \mathbb{N}$, occurs due

to discretizing the noise spatially, i.e. due to $\mathbb{E}\|W_t - W_t^K\|_H^2$ for $K \in \mathbb{N}$ and $t \in [0, T]$. If $U_0 \subset U$ is finite dimensional we choose $\mathcal{J}_K := \{j \in \mathcal{J} \mid \mu_j \neq 0\}$ for all $K \in \mathbb{N}$ and obtain $(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j)^\alpha = 0$ for every $K \in \mathbb{N}$ in that case. The third term, i.e. $M^{-\min(2(\gamma-\beta), \gamma)}$ for $M \in \mathbb{N}$, corresponds to the temporal discretization error and converges to zero as the number of time steps $M \in \mathbb{N}$ goes to infinity.

4 Examples

In this section Theorem 1 is illustrated with various examples. To this end let $d \in \{1, 2, 3\}$ and let $H = U = L^2((0, 1)^d, \mathbb{R})$ be the \mathbb{R} -Hilbert space of equivalence classes of $\mathcal{B}((0, 1)^d)/\mathcal{B}(\mathbb{R})$ -measurable and Lebesgue square integrable functions from $(0, 1)^d$ to \mathbb{R} . As usual we do not distinguish between a $\mathcal{B}((0, 1)^d)/\mathcal{B}(\mathbb{R})$ -measurable and Lebesgue square integrable function from $(0, 1)^d$ to \mathbb{R} and its equivalence class in H . The scalar product and the norm in H and U are given by

$$\langle v, w \rangle_H = \langle v, w \rangle_U = \int_{(0,1)^d} v(x) \cdot w(x) dx$$

and

$$\|v\|_H = \|v\|_U = \left(\int_{(0,1)^d} |v(x)|^2 dx \right)^{\frac{1}{2}}$$

for all $v, w \in H = U$. Additionally, the notations

$$\|v\|_{C((0,1)^d, \mathbb{R})} := \sup_{x \in (0,1)^d} |v(x)| \in [0, \infty]$$

and

$$\|v\|_{C^r((0,1)^d, \mathbb{R})} := \sup_{x \in (0,1)^d} |v(x)| + \sup_{\substack{x, y \in (0,1)^d \\ x \neq y}} \frac{|v(x) - v(y)|}{\|x - y\|_{\mathbb{R}^d}^r} \in [0, \infty]$$

are used throughout this section for all functions $v : (0, 1)^d \rightarrow \mathbb{R}$ and all $r \in (0, 1]$. Here and below we use the Euclidean norms $\|x\|_{\mathbb{R}^n} := (|x_1|^2 + \dots + |x_n|^2)^{1/2}$ for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and all $n \in \mathbb{N}$. Concerning the Wiener process $W : [0, T] \times \Omega \rightarrow U$ we assume in this section that the eigenfunctions $g_j \in U$, $j \in \mathcal{J}$, of the covariance operator $Q : U \rightarrow U$ are continuous and satisfy

$$\sup_{j \in \mathcal{J}} \|g_j\|_{C((0,1)^d, \mathbb{R})} < \infty \quad \text{and} \quad \sum_{j \in \mathcal{J}} \left(\mu_j \|g_j\|_{C^r((0,1)^d, \mathbb{R})}^2 \right) < \infty \quad (29)$$

for some $\rho \in (0, 1)$. We will give some concrete examples for the $(g_j)_{j \in \mathcal{J}}$ that fulfill (29) later. Additionally, we denote by $s_n(D) \in [0, \infty)$, $n \in \mathbb{N}$, the sequence of characteristic numbers of a compact operator $D : H \rightarrow H$ (see, e.g., Section 9 in Chapter XI in [9]). Finally, we define the Schatten norm by

$$\|D\|_{S_p(H)} := \left(\sum_{n=1}^{\infty} |s_n(D)|^p \right)^{\frac{1}{p}} \in [0, \infty]$$

for every compact operator $D : H \rightarrow H$ and every $p \in [1, \infty)$ (see also the above named reference).

We now present a prominent example of **the linear operator A in Assumption 1**. Let $\mathcal{I} = \mathbb{N}^d$ and let $e_i \in H$ for $i \in \mathcal{I}$ be given by

$$e_i(x) = 2^{\frac{d}{2}} \sin(i_1 \pi x_1) \cdot \dots \cdot \sin(i_d \pi x_d)$$

for all $x = (x_1, \dots, x_d) \in (0, 1)^d$ and all $i = (i_1, \dots, i_d) \in \mathbb{N}^d$. Additionally, let $\kappa \in (0, \infty)$ be a fixed real number and let $(\lambda_i)_{i \in \mathcal{I}}$ be given by

$$\lambda_i = \kappa \pi^2 ((i_1)^2 + \dots + (i_d)^2)$$

for all $i = (i_1, \dots, i_d) \in \mathbb{N}^d$. Hence, the linear operator $A : D(A) \subset H \rightarrow H$ in Assumption 1 reduces to the Laplacian with Dirichlet boundary conditions times the constant $\kappa \in (0, \infty)$, i.e.,

$$Av = \kappa \cdot \Delta v = \kappa \left(\left(\frac{\partial^2}{\partial x_1^2} \right) v + \dots + \left(\frac{\partial^2}{\partial x_d^2} \right) v \right)$$

holds for all $v \in D(A)$ in this section. Furthermore, let $(\mathcal{I}_N)_{N \in \mathbb{N}}$ be given by $\mathcal{I}_N = \{1, \dots, N\}^d$ for all $N \in \mathbb{N}$.

In order to formulate **the drift term in Assumption 2**, let $\beta = \frac{d}{5}$ and let $f : (0, 1)^d \times \mathbb{R} \rightarrow \mathbb{R}$ be a twice continuously differentiable function with $\int_{(0,1)^d} |f(x, 0)|^2 dx < \infty$ and $\sup_{x \in (0,1)^d} \sup_{y \in \mathbb{R}} \left| \left(\frac{\partial^n}{\partial y^n} f \right)(x, y) \right| < \infty$ for all $n \in \{1, 2\}$. Then the (in general nonlinear) operator $F : V_\beta \rightarrow H$ given by

$$(F(v))(x) = f(x, v(x))$$

for all $x \in (0, 1)^d$ and all $v \in V_\beta$ satisfies Assumption 2 since $V_\beta = V_{\frac{d}{5}} \subset L^5((0, 1)^d, \mathbb{R})$ continuously (see Remark 6.94 in [51]).

The formulation of **the diffusion term in Assumption 3** is more subtle. Let $b : (0, 1)^d \times \mathbb{R} \rightarrow \mathbb{R}$ be a twice continuously differentiable function with

$$|b(x, 0)| \leq q, \quad \left| \left(\frac{\partial^n}{\partial y^n} b \right)(x, y) \right| \leq q, \quad \left\| \left(\frac{\partial}{\partial x} b \right)(x, y) \right\|_{L(\mathbb{R}^d, \mathbb{R})} \leq q \quad (30)$$

and

$$\left| \left(\frac{\partial}{\partial y} b \right) (x, y) \cdot b(x, y) - \left(\frac{\partial}{\partial y} b \right) (x, z) \cdot b(x, z) \right| \leq q |y - z| \quad (31)$$

for all $x \in (0, 1)^d$, $y, z \in \mathbb{R}$, $n \in \{1, 2\}$ and some given $q \in (0, \infty)$. Then let $B : V_\beta \rightarrow HS(U_0, H)$ be the (in general nonlinear) operator

$$(B(v)u)(x) = b(x, v(x)) \cdot u(x) \quad (32)$$

for all $x \in (0, 1)^d$, $v \in V_\beta$ and all $u \in U_0 \subset U = H$. We now check step by step that $B : V_\beta \rightarrow HS(U_0, H)$ given by (32) satisfies Assumption 3. First of all, B is well defined. More precisely, we have

$$\begin{aligned} \|B(v)\|_{HS(U_0, H)}^2 &= \sum_{j \in \mathcal{J}} \|B(v) \sqrt{\mu_j} g_j\|_H^2 = \sum_{j \in \mathcal{J}} \mu_j \|B(v) g_j\|_H^2 \\ &= \sum_{j \in \mathcal{J}} \mu_j \left(\int_{(0,1)^d} |b(x, v(x)) \cdot g_j(x)|^2 dx \right) \\ &\leq \sum_{j \in \mathcal{J}} \mu_j \left(\int_{(0,1)^d} |b(x, v(x))|^2 dx \right) \left(\sup_{x \in (0,1)^d} |g_j(x)|^2 \right) \end{aligned}$$

and hence

$$\begin{aligned} &\|B(v)\|_{HS(U_0, H)}^2 \\ &\leq \sum_{j \in \mathcal{J}} \mu_j \left(\int_{(0,1)^d} (|b(x, v(x)) - b(x, 0)| + |b(x, 0)|)^2 dx \right) \|g_j\|_{C((0,1)^d, \mathbb{R})}^2 \\ &\leq \sum_{j \in \mathcal{J}} q^2 \mu_j \left(\int_{(0,1)^d} (|v(x)| + 1)^2 dx \right) \|g_j\|_{C((0,1)^d, \mathbb{R})}^2 \end{aligned}$$

and finally

$$\begin{aligned} \|B(v)\|_{HS(U_0, H)} &\leq q (\|v\|_H + 1) \left(\sum_{j \in \mathcal{J}} \mu_j \|g_j\|_{C((0,1)^d, \mathbb{R})}^2 \right)^{\frac{1}{2}} \\ &\leq q (\|v\|_H + 1) \left(\sum_{j \in \mathcal{J}} \mu_j \right)^{\frac{1}{2}} \left(\sup_{j \in \mathcal{J}} \|g_j\|_{C((0,1)^d, \mathbb{R})} \right) \quad (33) \\ &= q \sqrt{\text{Tr}(Q)} \left(\sup_{j \in \mathcal{J}} \|g_j\|_{C((0,1)^d, \mathbb{R})} \right) (\|v\|_H + 1) < \infty \end{aligned}$$

for every $v \in V_\beta$ which indeed shows that B is well defined. Moreover, B is twice continuously Fréchet differentiable and we have

$$\begin{aligned} \|B'(v)u\|_{HS(U_0, H)}^2 &= \sum_{j \in \mathcal{J}} \|(B'(v)u) \sqrt{\mu_j} g_j\|_H^2 = \sum_{j \in \mathcal{J}} \mu_j \|(B'(v)u) g_j\|_H^2 \\ &= \sum_{j \in \mathcal{J}} \mu_j \left(\int_{(0,1)^d} \left| \left(\frac{\partial}{\partial y} b \right) (x, v(x)) \cdot u(x) \cdot g_j(x) \right|^2 dx \right) \\ &\leq \sum_{j \in \mathcal{J}} q^2 \mu_j \left(\int_{(0,1)^d} |u(x) \cdot g_j(x)|^2 dx \right) \end{aligned}$$

and hence

$$\begin{aligned} \|B'(v)u\|_{HS(U_0, H)} &\leq \left(\sum_{j \in \mathcal{J}} q^2 \mu_j \|u\|_H^2 \|g_j\|_{C((0,1)^d, \mathbb{R})}^2 \right)^{\frac{1}{2}} \\ &\leq q \|u\|_H \left(\sum_{j \in \mathcal{J}} \mu_j \right)^{\frac{1}{2}} \left(\sup_{j \in \mathcal{J}} \|g_j\|_{C((0,1)^d, \mathbb{R})} \right) \\ &= q \sqrt{\text{Tr}(Q)} \left(\sup_{j \in \mathcal{J}} \|g_j\|_{C((0,1)^d, \mathbb{R})} \right) \|u\|_H \end{aligned}$$

for every $u, v \in V_\beta$ which shows

$$\sup_{v \in V_\beta} \|B'(v)\|_{L(H, HS(U_0, H))} \leq q \sqrt{\text{Tr}(Q)} \left(\sup_{j \in \mathcal{J}} \|g_j\|_{C((0,1)^d, \mathbb{R})} \right) < \infty.$$

Additionally, we have

$$\begin{aligned} \|B''(v)(u, w)\|_{HS(U_0, H)}^2 &= \sum_{j \in \mathcal{J}} \|B''(v)(u, w) \sqrt{\mu_j} g_j\|_H^2 \\ &= \sum_{j \in \mathcal{J}} \mu_j \left(\int_{(0,1)^d} \left| \left(\frac{\partial^2}{\partial y^2} b \right) (x, v(x)) \cdot u(x) \cdot w(x) \cdot g_j(x) \right|^2 dx \right) \\ &\leq \sum_{j \in \mathcal{J}} q^2 \mu_j \left(\int_{(0,1)^d} |u(x) \cdot w(x)|^2 dx \right) \|g_j\|_{C((0,1)^d, \mathbb{R})}^2 \end{aligned}$$

and using $L^5((0,1)^d, \mathbb{R}) \subset L^4((0,1)^d, \mathbb{R})$ continuously shows

$$\begin{aligned} & \|B''(v)(u, w)\|_{HS(U_0, H)} \\ & \leq q\sqrt{\text{Tr}(Q)} \left(\int_{(0,1)^d} |u(x)|^4 dx \right)^{\frac{1}{4}} \left(\int_{(0,1)^d} |w(x)|^4 dx \right)^{\frac{1}{4}} \left(\sup_{j \in \mathcal{J}} \|g_j\|_{C((0,1)^d, \mathbb{R})} \right) \\ & \leq q\sqrt{\text{Tr}(Q)} \left(\sup_{j \in \mathcal{J}} \|g_j\|_{C((0,1)^d, \mathbb{R})} \right) \left(\int_{(0,1)^d} |u(x)|^5 dx \right)^{\frac{1}{5}} \left(\int_{(0,1)^d} |w(x)|^5 dx \right)^{\frac{1}{5}} \end{aligned}$$

for every $u, v, w \in V_\beta$. Therefore, $V_\beta \subset L^5((0,1)^d, \mathbb{R})$ continuously shows

$$\sup_{v \in V_\beta} \|B''(v)\|_{L^2(V_\beta, HS(U_0, H))} < \infty$$

due to (29) and hence, it remains to establish (20)-(22) and the symmetry of $B'(v)B(v) \in HS^{(2)}(U_0, H)$ for all $v \in V_\beta$. For the latter one, note that

$$\begin{aligned} [(B'(v)B(v))(u, \tilde{u})](x) &= [B'(v)(B(v)u)\tilde{u}](x) \\ &= \left(\frac{\partial}{\partial y} b \right)(x, v(x)) \cdot b(x, v(x)) \cdot u(x) \cdot \tilde{u}(x) \quad (34) \end{aligned}$$

holds for all $x \in (0,1)^d$, $u, \tilde{u} \in U_0$ and all $v \in V_\beta$ which shows that $B'(v)B(v) \in HS^{(2)}(U_0, H)$ is indeed symmetric for all $v \in V_\beta$. Moreover, we have

$$\begin{aligned} & \|B'(v)B(v) - B'(w)B(w)\|_{HS^{(2)}(U_0, H)}^2 \\ &= \sum_{j, k \in \mathcal{J}} \mu_j \mu_k \|B'(v)(B(v)g_j)g_k - B'(w)(B(w)g_j)g_k\|_H^2 \\ &\leq \sum_{j, k \in \mathcal{J}} \mu_j \mu_k \left(\int_{(0,1)^d} \left| \left(\frac{\partial}{\partial y} b \right)(x, v(x)) \cdot b(x, v(x)) \right. \right. \\ &\quad \left. \left. - \left(\frac{\partial}{\partial y} b \right)(x, w(x)) \cdot b(x, w(x)) \right|^2 dx \right) \|g_j\|_{C((0,1)^d, \mathbb{R})}^2 \|g_k\|_{C((0,1)^d, \mathbb{R})}^2 \end{aligned}$$

and using (31) yields

$$\begin{aligned} & \|B'(v)B(v) - B'(w)B(w)\|_{HS^{(2)}(U_0, H)} \\ & \leq q \|v - w\|_H \left(\sum_{j, k \in \mathcal{J}} \mu_j \mu_k \right)^{\frac{1}{2}} \left(\sup_{i \in \mathcal{J}} \|g_i\|_{C((0,1)^d, \mathbb{R})}^2 \right) \\ &= q \text{Tr}(Q) \left(\sup_{i \in \mathcal{J}} \|g_i\|_{C((0,1)^d, \mathbb{R})}^2 \right) \|v - w\|_H \end{aligned}$$

for all $v, w \in V_\beta$ which shows that (21) indeed holds. Estimates (20) and (22) will be verified in the more concrete examples in Subsections 4.1-4.3 below.

Concerning **the initial value in Assumption 4**, let $x_0 : [0, 1]^d \rightarrow \mathbb{R}$ be a twice continuously differentiable function with $x_0|_{\partial(0,1)^d} \equiv 0$. Then the $\mathcal{F}_0/\mathcal{B}(V_\gamma)$ -measurable mapping $\xi : \Omega \rightarrow V_\gamma$ given by $\xi(\omega) = x_0$ for all $\omega \in \Omega$ fulfills Assumption 4 for all $\gamma \in (0, 1)$.

Having constructed examples of Assumptions 1-4, we now formulate the SPDE (23) in the setting of this section. More formally, in the setting above the SPDE (23) reduces to

$$dX_t(x) = \left[\kappa \Delta X_t(x) + f(x, X_t(x)) \right] dt + b(x, X_t(x)) dW_t(x) \quad (35)$$

with $X_t|_{\partial(0,1)^d} \equiv 0$ and $X_0(x) = x_0(x)$ for $t \in [0, T]$ and $x \in (0, 1)^d$. Moreover, we define a family $\beta^j : [0, T] \times \Omega \rightarrow \mathbb{R}$, $j \in \{k \in \mathcal{J} \mid \mu_k \neq 0\}$, of independent standard Brownian motions by

$$\beta_t^j(\omega) := \frac{1}{\sqrt{\mu_j}} \langle g_j, W_t(\omega) \rangle_U$$

for all $\omega \in \Omega$, $t \in [0, T]$ and all $j \in \mathcal{J}$ with $\mu_j \neq 0$. Using this notation, the SPDE (35) can be written as

$$dX_t(x) = \left[\kappa \Delta X_t(x) + f(x, X_t(x)) \right] dt + \sum_{\substack{j \in \mathcal{J} \\ \mu_j \neq 0}} \left[b(x, X_t(x)) \sqrt{\mu_j} g_j(x) \right] d\beta_t^j \quad (36)$$

with $X_t|_{\partial(0,1)^d} \equiv 0$ and $X_0(x) = x_0(x)$ for $t \in [0, T]$ and $x \in (0, 1)^d$.

The algorithm (27) applied to the SPDE (35) then reduces to $Y_0^{N,M,K} = P_N(x_0)$ and

$$\begin{aligned} Y_{m+1}^{N,M,K} = & P_N e^{A \frac{T}{M}} \left(Y_m^{N,M,K} + \frac{T}{M} \cdot f(\cdot, Y_m^{N,M,K}) + b(\cdot, Y_m^{N,M,K}) \cdot \Delta W_m^{M,K} \right. \\ & \left. + \frac{1}{2} \left(\frac{\partial}{\partial y} b \right) (\cdot, Y_m^{N,M,K}) \cdot b(\cdot, Y_m^{N,M,K}) \cdot \left((\Delta W_m^{M,K})^2 - \frac{T}{M} \sum_{j \in \mathcal{J}_K} \mu_j (g_j)^2 \right) \right) \end{aligned} \quad (37)$$

for all $m \in \{0, 1, \dots, M-1\}$ and all $N, M, K \in \mathbb{N}$. Finally, Theorem 1 shows the existence of a real number $C \in (0, \infty)$ such that

$$\begin{aligned} & \left(\mathbb{E} \int_{(0,1)^d} \left| X_T(x) - Y_M^{N,M,K}(x) \right|^2 dx \right)^{\frac{1}{2}} \\ & \leq C \left(N^{-2\gamma} + \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^\alpha + M^{-\min(2(\gamma-\beta), \gamma)} \right) \end{aligned} \quad (38)$$

holds for every $N, M, K \in \mathbb{N}$. We now illustrate estimate (38) in the following three more concrete examples. We begin with the introductory example from Section 1 (see (7) and (18)).

4.1 A one-dimensional stochastic reaction diffusion equation

In this subsection let $d = 1$, $T = 1$, $\kappa = \frac{1}{100}$, let $x_0 : [0, 1] \rightarrow \mathbb{R}$ be given by $x_0(x) = 0$ for all $x \in [0, 1]$, let $f, b : (0, 1) \times \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x, y) = 1 - y$ and $b(x, y) = \frac{1-y}{1+y^2}$ for all $x \in (0, 1)$, $y \in \mathbb{R}$, let $\mathcal{J} = \mathbb{N}$, let $\mathcal{J}_K = \{1, 2, \dots, K\}$ for all $K \in \mathbb{N}$, let $\mu_j = \frac{1}{j^2}$ and let $g_j = e_j$ for all $j \in \mathbb{N}$. The SPDE (36) thus reduces to

$$dX_t(x) = \left[\frac{1}{100} \frac{\partial^2}{\partial x^2} X_t(x) + 1 - X_t(x) \right] dt + \sum_{j=1}^{\infty} \frac{1 - X_t(x)}{1 + X_t(x)^2} \frac{\sqrt{2}}{j} \sin(j\pi x) d\beta_t^j \quad (39)$$

with $X_t(0) = X_t(1) = 0$ and $X_0(x) = 0$ for $x \in (0, 1)$ and $t \in [0, 1]$. The SPDE (39) is nothing else than equation (18) in the introduction. In order to apply Theorem 1 it remains to verify (20) and (22). Estimate (20) is fulfilled for all $\delta \in (0, \frac{1}{4})$ here due to Subsection 4.3 in [34]. In order to establish (22) several preparations are needed. More formally, let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be a further probability space on which a sequence $\chi_i : \tilde{\Omega} \rightarrow \mathbb{R}$, $i \in \mathbb{N}$, of $\tilde{\mathcal{F}}/\mathcal{B}(\mathbb{R})$ -measurable independent standard normal random variables is defined. Then we define the $\tilde{\mathcal{F}}/\mathcal{B}(U_0)$ -measurable mappings $\chi^{K, \vartheta} : \tilde{\Omega} \rightarrow U_0$ by

$$\chi^{K, \vartheta}(\omega, x) := \sum_{i=1}^K \chi_i(\omega) (\lambda_i)^{-\vartheta} e_i(x)$$

for all $\omega \in \tilde{\Omega}$, $x \in (0, 1)$, $K \in \mathbb{N}$ and all $\vartheta \in (0, \frac{1}{2})$. It will be essential to estimate $\mathbb{E} |\chi^{K, \vartheta}(x)|^2$ and $\mathbb{E} |\chi^{K, \vartheta}(x) - \chi^{K, \vartheta}(y)|^2$ for $x, y \in (0, 1)$, $K \in \mathbb{N}$ and $\vartheta \in (0, \frac{1}{2})$ in order to check (22). To this end note that

$$\begin{aligned} \mathbb{E} |\chi^{K, \vartheta}(x)|^2 &= \sum_{i=1}^K (\lambda_i)^{-2\vartheta} |e_i(x)|^2 \leq 2 \sum_{i=1}^K (\kappa \pi^2 i^2)^{-2\vartheta} \\ &\leq 2 (1 + \kappa^{-1}) \left(\sum_{i=1}^{\infty} i^{-4\vartheta} \right) \end{aligned} \quad (40)$$

holds for every $x \in (0, 1)$, $K \in \mathbb{N}$ and every $\vartheta \in (0, \frac{1}{2})$. Moreover, we have

$$\begin{aligned} \mathbb{E} |\chi^{K,\vartheta}(x) - \chi^{K,\vartheta}(y)|^2 &= \sum_{i=1}^K (\lambda_i)^{-2\vartheta} |e_i(x) - e_i(y)|^2 \\ &\leq \sum_{i=1}^K (\lambda_i)^{-2\vartheta} |e_i(x) - e_i(y)|^{2s} (|e_i(x)| + |e_i(y)|)^{2(1-s)} \\ &\leq \sum_{i=1}^K (\lambda_i)^{-2\vartheta} (2\pi^2 i^2)^s 8^{(1-s)} |x - y|^{2s} \end{aligned}$$

and

$$\begin{aligned} \mathbb{E} |\chi^{K,\vartheta}(x) - \chi^{K,\vartheta}(y)|^2 &\leq 8 \left(\sum_{i=1}^K (\kappa \pi^2 i^2)^{-2\vartheta} (\pi^2 i^2)^s \right) |x - y|^{2s} \\ &\leq \frac{8}{\kappa^{2\vartheta}} \left(\sum_{i=1}^K (\pi i)^{(2s-4\vartheta)} \right) |x - y|^{2s} \\ &\leq 3 (1 + \kappa^{-1}) \left(\sum_{i=1}^{\infty} i^{(2s-4\vartheta)} \right) |x - y|^{2s} \end{aligned} \tag{41}$$

for every $x, y \in (0, 1)$, $K \in \mathbb{N}$, $\vartheta \in (\frac{s}{2} + \frac{1}{4}, \frac{1}{2})$ and every $s \in (0, \frac{1}{2})$. We also use the notation

$$\|v\|_{W^{r,2}} := \left(\int_0^1 |v(x)|^2 dx + \int_0^1 \int_0^1 \frac{|v(x) - v(y)|^2}{|x - y|^{(1+2r)}} dx dy \right)^{\frac{1}{2}} \in [0, \infty]$$

for every $\mathcal{B}((0,1))/\mathcal{B}(\mathbb{R})$ -measurable mapping $v : (0,1) \rightarrow \mathbb{R}$ and every $r \in (0, \infty)$. Then we obtain

$$\begin{aligned}
& \mathbb{E} \|B(v)\chi^{K,\vartheta}\|_{W^{r,2}((0,1),\mathbb{R})}^2 \\
&= \int_0^1 \mathbb{E} |b(x, v(x)) \cdot \chi^{K,\vartheta}(x)|^2 dx \\
&\quad + \int_0^1 \int_0^1 \frac{\mathbb{E} |b(x, v(x)) \cdot \chi^{K,\vartheta}(x) - b(y, v(y)) \cdot \chi^{K,\vartheta}(y)|^2}{|x-y|^{(1+2r)}} dx dy \\
&\leq 2 \int_0^1 |b(x, v(x))|^2 \mathbb{E} |\chi^{K,\vartheta}(x)|^2 dx \\
&\quad + 2 \int_0^1 \int_0^1 \frac{|b(x, v(x))|^2 \mathbb{E} |\chi^{K,\vartheta}(x) - \chi^{K,\vartheta}(y)|^2}{|x-y|^{(1+2r)}} dx dy \\
&\quad + 2 \int_0^1 \int_0^1 \frac{|b(x, v(x)) - b(y, v(y))|^2 \mathbb{E} |\chi^{K,\vartheta}(y)|^2}{|x-y|^{(1+2r)}} dx dy
\end{aligned}$$

and using (40) and (41) shows

$$\begin{aligned}
& \mathbb{E} \|B(v)\chi^{K,\vartheta}\|_{W^{r,2}((0,1),\mathbb{R})}^2 \\
&\leq 4(1 + \kappa^{-1}) \left(\sum_{i=1}^{\infty} i^{-4\vartheta} \right) \|b(\cdot, v)\|_{W^{r,2}((0,1),\mathbb{R})}^2 \\
&\quad + 6(1 + \kappa^{-1}) \left(\sum_{i=1}^{\infty} i^{(2s-4\vartheta)} \right) \int_0^1 \int_0^1 \frac{|b(x, v(x))|^2}{|x-y|^{(1+2r-2s)}} dx dy \\
&\leq 4(1 + \kappa^{-1}) \left(\sum_{i=1}^{\infty} i^{(2s-4\vartheta)} \right) \|b(\cdot, v)\|_{W^{r,2}((0,1),\mathbb{R})}^2 \\
&\quad + 12(1 + \kappa^{-1}) \left(\sum_{i=1}^{\infty} i^{(2s-4\vartheta)} \right) \|b(\cdot, v)\|_H^2 \int_0^1 y^{(2s-2r-1)} dy \\
&\leq \frac{10(1 + \kappa^{-1})}{(s-r)} \left(\sum_{i=1}^{\infty} i^{(2s-4\vartheta)} \right) \|b(\cdot, v)\|_{W^{r,2}((0,1),\mathbb{R})}^2
\end{aligned}$$

for every $v \in H$, $\vartheta \in (\frac{s}{2} + \frac{1}{4}, \frac{1}{2})$, $s \in (r, \frac{1}{2})$, $K \in \mathbb{N}$ and every $r \in (0, \frac{1}{2})$. Therefore, inequality (23) in Section 4 in [34] gives

$$\begin{aligned}
& \left(\sup_{K \in \mathbb{N}} \mathbb{E} \|B(v) \chi^{K, \vartheta}\|_{W^{r, 2}((0, 1), \mathbb{R})}^2 \right)^{\frac{1}{2}} \\
& \leq \frac{4(1 + \kappa^{-1})}{(s - r)} \sqrt{\sum_{i=1}^{\infty} i^{(2s-4\vartheta)} \|b(\cdot, v)\|_{W^{r, 2}((0, 1), \mathbb{R})}^2} \\
& \leq \frac{4(1 + \kappa^{-1})}{(s - r)} \left(\sum_{i=1}^{\infty} i^{(2s-4\vartheta)} \right) \frac{3qC_{\frac{r}{2}}}{(1 - r)} (1 + \|v\|_{V_{\frac{r}{2}}}) \\
& \leq \frac{12C_{\frac{r}{2}}q(1 + \kappa^{-1})}{(s - r)^2} \left(\sum_{i=1}^{\infty} i^{(2s-4\vartheta)} \right) (1 + \|v\|_{V_{\frac{r}{2}}}) < \infty
\end{aligned} \tag{42}$$

for every $\vartheta \in (\frac{s}{2} + \frac{1}{4}, \frac{1}{2})$, $s \in (r, \frac{1}{2})$, $v \in V_{\frac{r}{2}}$ and every $r \in (0, \frac{1}{2})$. Moreover, we have

$$\begin{aligned}
\|(-A)^{-\vartheta} B(v) Q^{-\alpha}\|_{HS(U_0, H)} &= \|(-A)^{-\vartheta} B(v) Q^{(\frac{1}{2}-\alpha)}\|_{HS(H)} \\
&= \|Q^{(\frac{1}{2}-\alpha)} B(v) (-A)^{-\vartheta}\|_{HS(H)} \\
&= (\kappa\pi^2)^{(\frac{1}{2}-\alpha)} \left\| \left(\frac{Q}{\kappa\pi^2} \right)^{(\frac{1}{2}-\alpha)} B(v) (-A)^{-\vartheta} \right\|_{HS(H)} \\
&= (\kappa\pi^2)^{(\frac{1}{2}-\alpha)} \|B(v) (-A)^{-\vartheta}\|_{HS(H, V_{(\alpha-\frac{1}{2})})}
\end{aligned}$$

and using inequality (19) in Section 4 in [34] and estimate (42) in this article then yields

$$\begin{aligned}
& \|(-A)^{-\vartheta} B(v) Q^{-\alpha}\|_{HS(U_0, H)} \\
&= (\kappa\pi^2)^{(\frac{1}{2}-\alpha)} \left(\sup_{K \in \mathbb{N}} \mathbb{E} \|B(v) \chi^{K, \vartheta}\|_{V_{(\alpha-\frac{1}{2})}}^2 \right)^{\frac{1}{2}} \\
&\leq C_{(\alpha-\frac{1}{2})} (1 + \kappa^{-1}) \left(\sup_{K \in \mathbb{N}} \mathbb{E} \|B(v) \chi^{K, \vartheta}\|_{W^{2\alpha-1, 2}((0, 1), \mathbb{R})}^2 \right)^{\frac{1}{2}} \\
&\leq \frac{12C_{(\alpha-\frac{1}{2})}^2 q (1 + \kappa^{-1})^2}{(s + 1 - 2\alpha)^2} \left(\sum_{i=1}^{\infty} i^{(2s-4\vartheta)} \right) (1 + \|v\|_{V_{(\alpha-\frac{1}{2})}}) < \infty
\end{aligned}$$

for every $\vartheta \in (\frac{s}{2} + \frac{1}{4}, \frac{1}{2})$, $s \in (2\alpha - 1, \frac{1}{2})$, $v \in V_{(\alpha-\frac{1}{2})}$ and every $\alpha \in (\frac{1}{2}, \frac{3}{4})$. Therefore, estimate (22) is satisfied for all $\alpha \in (0, \frac{3}{4})$ and all $\gamma \in (\frac{1}{2}, \frac{3}{4})$. This

finally shows that Assumptions 1-4 are fulfilled for the SPDE (39) for all $\alpha \in (0, \frac{3}{4})$, $\beta = \frac{1}{5}$ and all $\gamma \in (\frac{1}{2}, \frac{3}{4})$.

Theorem 1 therefore yields the existence of real numbers $C_r \in (0, \infty)$, $r \in (0, \frac{3}{4})$, such that

$$\left(\mathbb{E} \int_0^1 \left| X_T(x) - Y_M^{N,M,K}(x) \right|^2 dx \right)^{\frac{1}{2}} \leq C_r \left(N^{(r-\frac{3}{2})} + K^{(r-\frac{3}{2})} + M^{(r-\frac{3}{4})} \right) \quad (43)$$

holds for all $N, M, K \in \mathbb{N}$ and all arbitrarily small $r \in (0, \frac{3}{4})$. In order to balance the error terms on the right hand side of (43) we choose $N^2 = K^2 = M$ in (43) and obtain the existence of real numbers $C_r \in (0, \infty)$, $r \in (0, \frac{3}{2})$, such that

$$\left(\mathbb{E} \int_0^1 \left| X_T(x) - Y_{N^2}^{N,N^2,N}(x) \right|^2 dx \right)^{\frac{1}{2}} \leq C_r \cdot N^{(r-\frac{3}{2})} \quad (44)$$

holds for all $N \in \mathbb{N}$ and all arbitrarily small $r \in (0, \frac{3}{2})$. Estimate (44) is nothing else than inequality (16) in the introduction. We also refer to Figure 1 in the introduction for numerical results illustrating (44).

4.2 A one-dimensional stochastic reaction diffusion equation with $AQ \neq QA$

In Subsection 4.1 we assumed that the eigenfunctions of the dominating linear operator A and of the covariance operator Q of the driving Wiener process $W : [0, T] \times \Omega \rightarrow H$ of the SPDE (35) coincide and in particular, we assumed in Subsection 4.1 that

$$AQv = QAv \quad (45)$$

holds for all $v \in D(A)$. However, our general setting in Section 2 does not need condition (45) to be fulfilled. To illustrate this fact we consider in this subsection an example in which (45) fails to hold. More formally, in this subsection let $d = 1$, $T = 1$, $\kappa = \frac{1}{20}$, let $x_0 : [0, 1] \rightarrow \mathbb{R}$ be given by $x_0(x) = 0$ for all $x \in [0, 1]$, let $f, b : (0, 1) \times \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x, y) = 1 - y$ and $b(x, y) = \frac{y}{1+y^2}$ for all $x \in (0, 1)$, $y \in \mathbb{R}$, let $\mathcal{J} = \{0, 1, 2, \dots\}$, let $\mathcal{J}_K = \{0, 1, \dots, K\}$ for all $K \in \mathbb{N}$, let $\mu_0 = 0$, $\mu_j = \frac{1}{j^3}$ and let $g_j : (0, 1) \rightarrow \mathbb{R}$ be given by $g_0(x) = 1$, $g_j(x) = \sqrt{2} \cos(j\pi x)$ for all $x \in (0, 1)$ and all $j \in \mathbb{N}$. The SPDE (35) thus reduces to

$$dX_t(x) = \left[\frac{1}{20} \frac{\partial^2}{\partial x^2} X_t(x) + 1 - X_t(x) \right] dt + \frac{X_t(x)}{1 + X_t(x)^2} dW_t(x) \quad (46)$$

with $X_t(0) = X_t(1) = 0$ and $X_0(x) = 0$ for $x \in (0, 1)$ and $t \in [0, 1]$. Of course, the SPDE (46) can also be written as

$$dX_t(x) = \left[\frac{1}{20} \frac{\partial^2}{\partial x^2} X_t(x) + 1 - X_t(x) \right] dt + \sum_{j=1}^{\infty} \frac{X_t(x)}{1 + X_t(x)^2} \frac{\sqrt{2}}{j^{1.5}} \cos(j\pi x) d\beta_t^j$$

with $X_t(0) = X_t(1) = 0$ and $X_0(x) = 0$ for $x \in (0, 1)$ and $t \in [0, 1]$. Estimate (20) is here fulfilled for all $\delta \in (0, \frac{1}{2})$ due to Subsection 4.2 in [34]. Moreover, as in Subsection 4.1 it can be shown that inequality (22) holds for all $\alpha \in (0, \frac{2}{3})$ and all $\gamma \in (\frac{1}{2}, 1)$. This finally shows that Assumptions 1-4 are fulfilled for the SPDE (46) for all $\alpha \in (0, \frac{2}{3})$, $\beta = \frac{1}{5}$ and all $\gamma \in (\frac{1}{2}, 1)$.

Theorem 1 therefore yields the existence of real numbers $C_r \in (0, \infty)$, $r \in (0, 1)$, such that

$$\left(\mathbb{E} \int_0^1 \left| X_T(x) - Y_M^{N,M,K}(x) \right|^2 dx \right)^{\frac{1}{2}} \leq C_r (N^{(r-2)} + K^{(r-2)} + M^{(r-1)}) \quad (47)$$

holds for all $N, M, K \in \mathbb{N}$ and all arbitrarily small $r \in (0, 1)$. Choosing $N^2 = K^2 = M$ in (47) hence gives the existence of real numbers $C_r \in (0, \infty)$, $r \in (0, 2)$, such that

$$\left(\mathbb{E} \int_0^1 \left| X_T(x) - Y_{N^2}^{N,N^2,N}(x) \right|^2 dx \right)^{\frac{1}{2}} \leq C_r \cdot N^{(r-2)} \quad (48)$$

holds for all $N \in \mathbb{N}$ and all arbitrarily small $r \in (0, 2)$. The approximation $Y_{N^2}^{N,N^2,N}$ thus converges in the root mean square sense to X_T with order 2— as N goes to infinity. Since $P_N(H) \subset H$ is N -dimensional and since N^2 time steps are used to simulate $Y_{N^2}^{N,N^2,N}$, $O(N^3 \log(N))$ computational operations and random variables are needed to simulate $Y_{N^2}^{N,N^2,N}$ here. Combining the computational effort $O(N^3 \log(N))$ and the convergence order 2— in (48) shows that the algorithm (37) in this article with $N^2 = K^2 = M$ needs about $O(\varepsilon^{-\frac{3}{2}})$ computational operations and random variables to achieve a root mean square precision $\varepsilon > 0$.

The linear implicit Euler scheme combined with spectral Galerkin methods which we denote by $\mathcal{F}/\mathcal{B}(H)$ -measurable mappings $Z_n^N : \Omega \rightarrow H$, $n \in \{0, 1, \dots, N^4\}$, $N \in \mathbb{N}$, is given by $Z_0^N = 0$ and

$$Z_{n+1}^N = P_N \left(I - \frac{T}{N^4} A \right)^{-1} \left(Z_n^N + \frac{T}{N^4} \cdot f(\cdot, Z_n^N) + b(\cdot, Z_n^N) \cdot \Delta W_n^{N^4,N} \right) \quad (49)$$

for all $n \in \{0, 1, \dots, N^4 - 1\}$ and all $N \in \mathbb{N}$ here.

In Figure 4 the root mean square approximation error $(\mathbb{E}\|X_T - Z_{N^4}^N\|_H^2)^{\frac{1}{2}}$ of the linear implicit Euler approximation $Z_{N^4}^N$ (see (49)) and the root mean square approximation error $(\mathbb{E}\|X_T - Y_{N^2}^{N,N^2,N}\|_H^2)^{\frac{1}{2}}$ of the approximation $Y_{N^2}^{N,N^2,N}$ in this article (see (27) and (37)) is plotted against the precise number of independent standard normal random variables needed to compute the corresponding approximation for $N \in \{4, 8, 16, 32\}$.

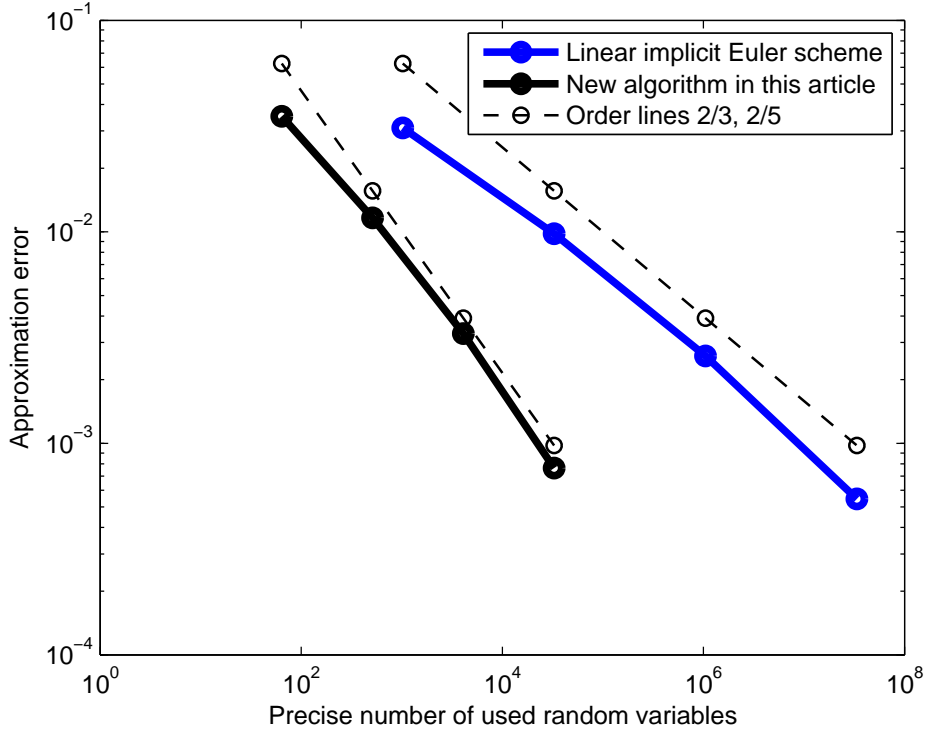


Figure 4: SPDE (46): Root mean square approximation error $(\mathbb{E}\|X_T - Z_{N^4}^N\|_H^2)^{\frac{1}{2}}$ of the linear implicit Euler approximation $Z_{N^4}^N$ (see (49)) and root mean square approximation error $(\mathbb{E}\|X_T - Y_{N^2}^{N,N^2,N}\|_H^2)^{\frac{1}{2}}$ of the approximation $Y_{N^2}^{N,N^2,N}$ in this article (see (27) and (37)) against the precise number of independent standard normal random variables needed to compute the corresponding approximation for $N \in \{4, 8, 16, 32\}$.

4.3 A two-dimensional stochastic heat equation and splitting-up approximations

In comparison to existing algorithms and approximation results in the literature the main contribution of this article is to break the computational complexity for SPDEs with possibly nonlinear diffusion and drift coefficients. However, in order to compare the Milstein type algorithm (27) in this article with previously considered numerical methods we consider a linear stochastic heat equation in this subsection. More formally, in this subsection let $d = 2$, $T = 1$, $\kappa = \frac{1}{50}$, let $x_0 : [0, 1]^2 \rightarrow \mathbb{R}$ be given by $x_0(x_1, x_2) = 2 \sin(\pi x_1) \sin(\pi x_2)$ for all $x_1, x_2 \in [0, 1]$, let $f, b : (0, 1)^2 \times \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x_1, x_2, y) = 0$ and $b(x_1, x_2, y) = y$ for all $x_1, x_2 \in (0, 1)$, $y \in \mathbb{R}$, let $\mathcal{J} = \mathbb{N}^2$, let $\mathcal{J}_K = \{1, 2, \dots, K\}^2$ for all $K \in \mathbb{N}$, let $\mu_{(j_1, j_2)} = (j_1 + j_2)^{-4}$ and let $g_{(j_1, j_2)} = e_{(j_1, j_2)}$ for all $j_1, j_2 \in \mathbb{N}$. The SPDE (35) thus reduces to

$$dX_t(x_1, x_2) = \left[\frac{1}{50} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) X_t(x_1, x_2) \right] dt + X_t(x_1, x_2) dW_t(x_1, x_2) \quad (50)$$

with $X_t|_{\partial(0,1)^2} \equiv 0$ and $X_0(x_1, x_2) = 2 \sin(\pi x_1) \sin(\pi x_2)$ for $x_1, x_2 \in (0, 1)$ and $t \in [0, 1]$. In view of (36) the SPDE (50) can also be written as

$$dX_t(x_1, x_2) = \left[\frac{1}{50} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) X_t(x_1, x_2) \right] dt + \sum_{j_1, j_2=1}^{\infty} \frac{X_t(x_1, x_2)}{(j_1 + j_2)^2} 2 \sin(j_1 \pi x_1) \sin(j_2 \pi x_2) d\beta_t^{(j_1, j_2)}$$

with $X_t|_{\partial(0,1)^2} \equiv 0$ and $X_0(x_1, x_2) = 2 \sin(\pi x_1) \sin(\pi x_2)$ for $x_1, x_2 \in (0, 1)$ and $t \in [0, 1]$.

Due to Subsection 4.3 in [34] inequality (20) holds for all $\delta \in (0, \frac{1}{2})$ here. In order to verify (22) the notation

$$\|v\|_{L^\infty((0,1)^2, \mathbb{R})} := \inf \left\{ R \in [0, \infty) \mid \lambda(\{x \in (0, 1)^2 \mid v(x) > R\}) = 0 \right\} \in [0, \infty]$$

is used for all $\mathcal{B}((0, 1)^2)/\mathcal{B}(\mathbb{R})$ -measurable mappings $v : (0, 1)^2 \rightarrow \mathbb{R}$ in this subsection. Then we obtain

$$\begin{aligned} \|v\|_{L^\infty((0,1)^2, \mathbb{R})} &\leq \sum_{i \in \mathbb{N}^2} |\langle e_i, v \rangle_H| \|e_i\|_{C((0,1)^2, \mathbb{R})} \\ &\leq 2 \left(\sum_{i \in \mathbb{N}^2} (\lambda_i)^{-2r} \right)^{\frac{1}{2}} \left(\sum_{i \in \mathbb{N}^2} (\lambda_i)^{2r} |\langle e_i, v \rangle_H|^2 \right)^{\frac{1}{2}} = 2 \left(\sum_{i \in \mathbb{N}^2} (\lambda_i)^{-2r} \right)^{\frac{1}{2}} \|v\|_{V_r} \end{aligned} \quad (51)$$

for all $v \in V_r$ and all $r \in (\frac{1}{2}, \infty)$. Moreover, we have

$$\begin{aligned}
& \left\| (-A)^{-\vartheta} B(v) Q^{-\alpha} \right\|_{HS(U_0, H)} \\
&= \left\| (-A)^{-\vartheta} B(v) Q^{(\frac{1}{2}-\alpha)} \right\|_{HS(H)} = \left\| (-A)^{-\vartheta} B(v) Q^{(\frac{1}{2}-\alpha)} \right\|_{S_2(H)} \\
&\leq \left\| (-A)^{-\vartheta} \right\|_{S_{\frac{1}{\alpha}}(H)} \|B(v)\|_{L(H)} \left\| Q^{(\frac{1}{2}-\alpha)} \right\|_{S_{\frac{2}{(1-2\alpha)}}(H)} \\
&\leq \left(\sum_{i \in \mathbb{N}^2} (\lambda_i)^{-\frac{\vartheta}{\alpha}} \right)^{\alpha} \|b(\cdot, v)\|_{L^\infty((0,1)^2, \mathbb{R})} \left(\sum_{j \in \mathcal{J}} \mu_j^{(\frac{1}{2}-\alpha)\frac{2}{(1-2\alpha)}} \right)^{\frac{(1-2\alpha)}{2}}
\end{aligned}$$

and using (51) shows

$$\begin{aligned}
& \left\| (-A)^{-\vartheta} B(v) Q^{-\alpha} \right\|_{HS(U_0, H)} \\
&\leq \left(\sum_{i \in \mathbb{N}^2} (\lambda_i)^{-\frac{\vartheta}{\alpha}} \right)^{\alpha} \left(q \|v\|_{L^\infty((0,1)^2, \mathbb{R})} + q \right) (\text{Tr}(Q))^{(\frac{1}{2}-\alpha)} \\
&\leq q (1 + \text{Tr}(Q)) \left(\sum_{i \in \mathbb{N}^2} (\lambda_i)^{-\frac{\vartheta}{\alpha}} \right)^{\alpha} \left(1 + \|v\|_{L^\infty((0,1)^2, \mathbb{R})} \right) \\
&\leq 2q (1 + \text{Tr}(Q)) \left(\sum_{i \in \mathbb{N}^2} (\lambda_i)^{-\frac{\vartheta}{\alpha}} \right)^{\alpha} \left(1 + \sum_{i \in \mathbb{N}^2} (\lambda_i)^{-2\gamma} \right) \left(1 + \|v\|_{V_\gamma} \right) < \infty
\end{aligned}$$

for every $\vartheta \in (\alpha, \frac{1}{2})$, $\alpha \in (0, \frac{1}{2})$, $v \in V_\gamma$ and every $\gamma \in (\frac{1}{2}, 1)$. Inequality (22) thus holds for all $\alpha \in (0, \frac{1}{2})$ and all $\gamma \in (\frac{1}{2}, 1)$ here. This finally shows that Assumptions 1-4 are fulfilled for the SPDE (50) for all $\alpha \in (0, \frac{1}{2})$, $\beta = \frac{2}{5}$ and all $\gamma \in (\frac{1}{2}, 1)$.

Theorem 1 therefore yields the existence of real numbers $C_r \in (0, \infty)$, $r \in (0, 1)$, such that

$$\begin{aligned}
& \left(\mathbb{E} \int_0^1 \int_0^1 \left| X_T(x_1, x_2) - Y_M^{N, M, K}(x_1, x_2) \right|^2 dx_1 dx_2 \right)^{\frac{1}{2}} \\
&\leq C_r \left(N^{(r-2)} + K^{(r-2)} + M^{(r-1)} \right) \quad (52)
\end{aligned}$$

holds for all $N, M, K \in \mathbb{N}$ and all arbitrarily small $r \in (0, 1)$. In order to balance the error terms on the right hand side of (52) we choose $M = N^2 = K^2$ in (52) and obtain the existence of real numbers $C_r \in (0, \infty)$, $r \in (0, 2)$, such that

$$\left(\mathbb{E} \int_0^1 \int_0^1 \left| X_T(x_1, x_2) - Y_{N^2}^{N, N^2, N}(x_1, x_2) \right|^2 dx_1 dx_2 \right)^{\frac{1}{2}} \leq C_r \cdot N^{(r-2)} \quad (53)$$

holds for all $N \in \mathbb{N}$ and all arbitrarily small $r \in (0, 2)$. The approximation $Y_{N^2}^{N, N^2, N}$ thus converges in the root mean square sense to X_T with order 2- as N goes to infinity. The numerical approximations $Y_n^{N, N^2, N} : \Omega \rightarrow H$, $n \in \{0, 1, \dots, N^2\}$, $N \in \mathbb{N}$, (see (27) and (37)) are here given by $Y_0^{N, N^2, N} = x_0$ and

$$Y_{n+1}^{N, N^2, N} = P_N e^{A \frac{T}{N^2}} \left(\left[1 + \Delta W_n^{N^2, N} + \frac{1}{2} \left(\Delta W_n^{N^2, N} \right)^2 - \frac{T}{2N^2} \sum_{j \in \mathcal{J}_K} \mu_j (g_j)^2 \right] \cdot Y_n^{N, N^2, N} \right) \quad (54)$$

for all $n \in \{0, 1, \dots, N^2 - 1\}$ and all $N \in \mathbb{N}$. Since $P_N(H) \subset H$ in N^2 -dimensional here and since N^2 time steps are used to simulate $Y_{N^2}^{N, N^2, N}$, $O(N^4 \log(N))$ computational operations and random variables are needed to simulate $Y_{N^2}^{N, N^2, N}$. Combining the computational effort $O(N^4 \log(N))$ and the convergence order 2- in (53) shows that the algorithm (54) in this article needs about $O(\varepsilon^{-2})$ computational operations and random variables to achieve a root mean square precision $\varepsilon > 0$.

The linear implicit Euler scheme combined with spectral Galerkin methods which we denote by $\mathcal{F}/\mathcal{B}(H)$ -measurable mappings $Z_n^N : \Omega \rightarrow H$, $n \in \{0, 1, \dots, N^4\}$, $N \in \mathbb{N}$, is given by $Z_0^N = x_0$ and

$$Z_{n+1}^N = P_N \left(I - \frac{T}{N^4} A \right)^{-1} \left(\left[1 + \Delta W_n^{N^4, N} \right] \cdot Z_n^N \right) \quad (55)$$

for all $n \in \{0, 1, \dots, N^4 - 1\}$ and all $N \in \mathbb{N}$ here.

Moreover, since the SPDE (50) is linear here, the splitting-up method in [20] can be used in order to solve (50) approximatively. The key idea of the splitting-up approach is to split the SPDE (50) into the explicit solvable subequations

$$d\tilde{X}_t(x_1, x_2) = \left[\frac{1}{50} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \tilde{X}_t(x_1, x_2) \right] dt, \quad \tilde{X}_t|_{\partial(0,1)^2} \equiv 0 \quad (56)$$

and

$$d\tilde{\tilde{X}}_t(x_1, x_2) = \tilde{\tilde{X}}_t(x_1, x_2) dW_t(x_1, x_2) \quad (57)$$

for $t \in [0, 1]$ and $x_1, x_2 \in (0, 1)$. For the solution processes $\tilde{X}, \tilde{\tilde{X}} : [0, T] \times \Omega \rightarrow H$ of (56) and (57) we obtain $\tilde{X}_t = e^{At} \tilde{X}_0$ and $\tilde{\tilde{X}}_t = e^{(W_t - \frac{t}{2} \sum_{j \in \mathcal{J}} \mu_j (g_j)^2)} \cdot \tilde{\tilde{X}}_0$ \mathbb{P} -a.s. for all $t \in [0, 1]$. This suggests the splitting-up approximation

$$X_t \approx e^{At} \left(e^{(W_t - \frac{t}{2} \sum_{j \in \mathcal{J}} \mu_j (g_j)^2)} \cdot X_0 \right)$$

for $t \in [0, 1]$ where $X : [0, T] \times \Omega \rightarrow H$ is the solution process of the SPDE (50). The resulting splitting-up method which we denote by $\mathcal{F}/\mathcal{B}(H)$ -measurable mappings $\tilde{Z}_n^N : \Omega \rightarrow H$, $n \in \{0, 1, \dots, N^2\}$, $N \in \mathbb{N}$, is then given by $\tilde{Z}_0^N = x_0$ and

$$\tilde{Z}_{n+1}^N = P_N e^{A \frac{T}{N^2}} \left(e^{\left(\Delta W_n^{N^2, N} - \frac{T}{2N^2} \sum_{j \in \mathcal{J}_K} \mu_j(g_j)^2 \right)} \cdot \tilde{Z}_n^N \right) \quad (58)$$

for all $n \in \{0, 1, \dots, N^2 - 1\}$ and all $N \in \mathbb{N}$. We remark that I. Göngy and N. Krylov considered in [20] temporal splitting-up approximations and we added an appropriate spatial approximation here in order to compare the splitting-up method with the algorithm (54) in this article. Using the Talyor approximation $e^x \approx 1 + x + \frac{x^2}{2}$ for all $x \in \mathbb{R}$ then yields

$$\begin{aligned} & e^{\left(\Delta W_n^{N^2, N} - \frac{T}{2N^2} \sum_{j \in \mathcal{J}_K} \mu_j(g_j)^2 \right)} \\ & \approx 1 + \Delta W_n^{N^2, N} - \frac{T}{2N^2} \sum_{j \in \mathcal{J}_K} \mu_j(g_j)^2 + \frac{1}{2} \left(\Delta W_n^{N^2, N} - \frac{T}{2N^2} \sum_{j \in \mathcal{J}_K} \mu_j(g_j)^2 \right)^2 \\ & \approx 1 + \Delta W_n^{N^2, N} + \frac{1}{2} \left(\Delta W_n^{N^2, N} \right)^2 - \frac{T}{2N^2} \sum_{j \in \mathcal{J}_K} \mu_j(g_j)^2 \end{aligned} \quad (59)$$

for all $n \in \{0, 1, \dots, N^2 - 1\}$ and all $N \in \mathbb{N}$. Using approximation (59) in (58) finally shows

$$\tilde{Z}_{n+1}^N \approx P_N e^{A \frac{T}{N^2}} \left(\left[1 + \Delta W_n^{N^2, N} + \frac{1}{2} \left(\Delta W_n^{N^2, N} \right)^2 - \frac{T}{2N^2} \sum_{j \in \mathcal{J}_K} \mu_j(g_j)^2 \right] \cdot \tilde{Z}_n^N \right)$$

for all $n \in \{0, 1, \dots, N^2 - 1\}$ and all $N \in \mathbb{N}$ which is nothing else than the recursion for $Y_n^{N, N^2, N}$, $n \in \{0, 1, \dots, N^2\}$, $N \in \mathbb{N}$ in (54). So, in the case of the linear SPDE (50) an alternative way for deriving the algorithm (54) in this article is to apply an appropriate Taylor approximation for the exponential function (see (59) for details) to the splitting-up method (58). More results for the splitting-up method can be found in [3, 4, 12, 19, 20, 21, 22, 23] and the references therein.

In Figure 5 the root mean square approximation error $(\mathbb{E} \|X_T - Z_{N^4}^N\|_H^2)^{\frac{1}{2}}$ of the linear implicit Euler approximation $Z_{N^4}^N$ (see (55)), the root mean square approximation error $(\mathbb{E} \|X_T - Y_{N^2}^{N, N^2, N}\|_H^2)^{\frac{1}{2}}$ of the approximation $Y_{N^2}^{N, N^2, N}$ in this article (see (54)) and the root mean square approximation error $(\mathbb{E} \|X_T - \tilde{Z}_{N^2}^N\|_H^2)^{\frac{1}{2}}$ of the splitting-up approximation $\tilde{Z}_{N^2}^N$ (see (58)) is plotted against the precise number of independent standard normal random variables needed

to compute the corresponding approximation for $N \in \{2, 4, 8, 16, 32\}$: It turns out that $Z_{32^4}^{32}$ ($32^6 = 1\,073\,741\,824$ random variables) in the case of the linear implicit Euler scheme (55), that $Y_{32}^{32,32^2,32}$ ($32^4 = 1\,048\,576$ random variables) in the case of the algorithm (54) in this article and that $\tilde{Z}_{32^2}^{32}$ ($32^4 = 1\,048\,576$ random variables) in the case of the splitting-up method (58) achieve a root mean square precision $\varepsilon = \frac{1}{1000}$ for the SPDE (50). The MATLAB codes for simulating $Z_{32^4}^{32}$ via (55), $Y_{32}^{32,32^2,32}$ via (54) and $\tilde{Z}_{32^2}^{32}$ via (58) are presented below in Figures 6, 7 and 8 respectively. The MATLAB code in Figure 6 requires on an INTEL PENTIUM D a CPU time of about **29 minutes and 57.06 seconds** (1797.06 seconds), the code in Figure 7 requires a CPU time of about **1.99 seconds** while the code in Figure 8 requires a CPU time of about **2.10 seconds** to be evaluated on the same computer.

Finally, we conclude that for linear SPDEs the splitting-up method converges with the same order as the algorithm (27) in this article. For SPDEs with nonlinear diffusion coefficients the splitting-up method can in general not be used efficiently anymore since the splitted subequations do in general not simplify in comparison to the original considered SPDE. In contrast to the splitting-up method the algorithm (27) works quite well for nonlinear equations as demonstrated in Section 1, Subsection 4.1, Subsection 4.2 and Section 5.

5 A further numerical example

Although the setting in Section 2 requires the nonlinear coefficients F and B of the SPDE (23) to be globally Lipschitz continuous, we strongly believe that our method (27) produces efficient results for SPDEs with non-globally Lipschitz nonlinearities. To substantiate this claim, we apply in this section the algorithm (27) and the linear implicit Euler scheme to a stochastic Burgers equation whose nonlinear drift term F is quadratic and therefore not globally Lipschitz continuous anymore. More formally, we consider the SPDE

$$dX_t(x) = \left[\frac{1}{100} \frac{\partial^2}{\partial x^2} X_t(x) - X_t(x) \left(\frac{\partial}{\partial x} X_t(x) \right) \right] dt + X_t(x) dW_t \quad (60)$$

with $X_t(0) = X_t(1) = 0$ and $X_0(x) = \frac{3\sqrt{2}}{5} (\sin(\pi x) + \sin(2\pi x))$ for $x \in (0, 1)$ and $t \in [0, T]$ on $H = L^2((0, 1), \mathbb{R})$ where $T = 1$ and where $W : [0, T] \times \Omega \rightarrow H$ is a standard Q -Wiener process with the covariance operator $Q : H \rightarrow H$ given by $(Qv)(x) = \sum_{n=1}^{\infty} n^{-3} \sin(n\pi x) \int_0^1 \sin(n\pi y) v(y) dy$ for all $x \in (0, 1)$

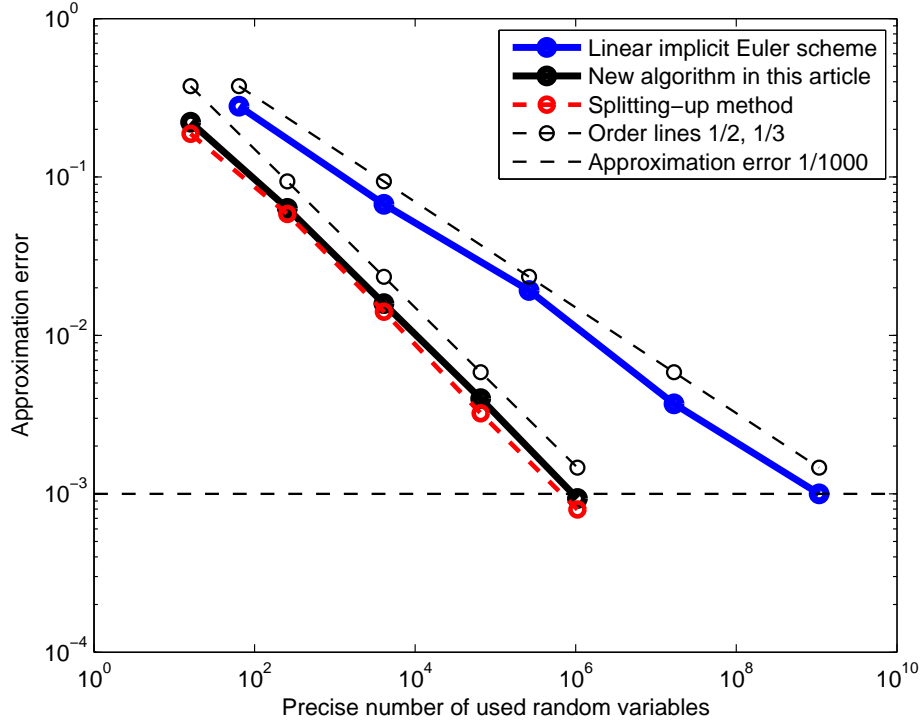


Figure 5: SPDE (50): Root mean square approximation error $(\mathbb{E}\|X_T - Z_{N^4}^N\|_H^2)^{\frac{1}{2}}$ of the linear implicit Euler approximation $Z_{N^4}^N$ (see (55)), root mean square approximation error $(\mathbb{E}\|X_T - Y_{N^2}^{N,N^2,N}\|_H^2)^{\frac{1}{2}}$ of the approximation $Y_{N^2}^{N,N^2,N}$ in this article (see (54)) and root mean square approximation error $(\mathbb{E}\|X_T - \tilde{Z}_{N^2}^N\|_H^2)^{\frac{1}{2}}$ of the splitting-up approximation $\tilde{Z}_{N^2}^N$ (see (58)) against the precise number of independent standard normal random variables needed to compute the corresponding approximation for $N \in \{2, 4, 8, 16, 32\}$.

```

1 N = 32; M = N^4; Y = zeros(N,N); Y(1,1) = 1;
2 [n1,n2] = meshgrid(1:N);
3 A = -(n1.^2 + n2.^2) * pi^2/50;
4 mu = 1./(n1.^4 + n2.^4);
5 for m=1:M
6     y = dst( dst( Y )' )' * 2;
7     dW = dst( dst( randn(N,N) .* sqrt(mu/M) )' )' * 2;
8     y = (1 + dW).* y;
9     Y = idst( idst( y' )' ) / 2 ./ ( 1 - A/M );
10 end
11 surf( n1/(N+1), n2/(N+1), dst( dst( Y )' )' * 2);

```

Figure 6: MATLAB code for simulating the linear implicit Euler approximation $Z_{N^4}^N$ with $N = 32$ (see (55)) for the SPDE (46).

```

1 N = 32; M = N^2; Y = zeros(N,N); Y(1,1) = 1;
2 [n1,n2] = meshgrid(1:N);
3 A = -(n1.^2 + n2.^2) * pi^2/50;
4 mu = 1./(n1.^4 + n2.^4);
5 g = zeros(N,N); grid = (1:N)/(N+1);
6 for m1=1:N
7     for m2=1:N
8         g = g+4*sin(m1*grid*pi).^2*sin(m2*grid*pi).^2.*mu(m1,m2)/M;
9     end
10 end
11 for m=1:M
12     y = dst( dst( Y )' )' * 2;
13     dW = dst( dst( randn(N,N) .* sqrt(mu/M) )' )' * 2;
14     y = (1 + dW + (dW.^2 - g)/2) .* y;
15     Y = exp( A/M ) .* idst( idst( y' )' ) / 2;
16 end
17 surf( n1/(N+1), n2/(N+1), dst( dst( Y )' )' * 2);

```

Figure 7: MATLAB code for simulating the approximation $Y_{N^2}^{N,N^2,N}$ in this article with $N = 32$ (see (54)) for the SPDE (46).

```

1 N = 32; M = N^2; Y = zeros(N,N); Y(1,1) = 1;
2 [n1,n2] = meshgrid(1:N);
3 A = -(n1.^2 + n2.^2) * pi^2/50;
4 mu = 1./(n1.^4 + n2.^4);
5 g = zeros(N,N); grid = (1:N)/(N+1);
6 for m1=1:N
7     for m2=1:N
8         g = g+4*sin(m1*grid*pi).^2*sin(m2*grid*pi).^2.*mu(m1,m2)/M;
9     end
10 end
11 for m=1:M
12     y = dst( dst( Y )' )' * 2;
13     dW = dst( dst( randn(N,N) .* sqrt(mu/M) )' )' * 2;
14     y = exp(dW - g/2).*y;
15     Y = exp( A/M ) .* idst( idst( y' )' ) / 2;
16 end
17 surf( n1/(N+1), n2/(N+1), dst( dst( Y )' )' * 2);

```

Figure 8: MATLAB code for simulating the splitting-up approximation $\tilde{Z}_{N^2}^N$ with $N = 32$ (see (58)) for the SPDE (46).

and all $v \in H$. The family $g_j \in H$, $j \in \mathcal{J}$, of functions with $\mathcal{J} = \mathbb{N}$ and $g_j(x) = \sqrt{2} \sin(j\pi x)$ for all $x \in (0, 1)$, $j \in \mathbb{N}$, thus represents an orthonormal basis of eigenfunctions of $Q : H \rightarrow H$. The corresponding eigenvalues $(\mu_j)_{j \in \mathcal{J}}$ satisfy $\mu_j = \frac{1}{j^2}$ for all $j \in \mathbb{N}$ and we choose $\mathcal{J}_K = \{1, 2, \dots, K\}$ for all $K \in \mathbb{N}$ here. Theoretical results for the stochastic Burgers equation including existence and uniqueness results can be found in [6, 10, 11], for instance.

The method (27) with $N^2 = K^2 = M$ applied to the SPDE (60) is given by $Y_0^{N,N^2,N} = P_N(X_0)$ and

$$\begin{aligned}
Y_{n+1}^{N,N^2,N} = & P_N e^{A \frac{T}{N^2}} \left(Y_n^{N,N^2,N} - \frac{T}{N^2} \cdot Y_n^{N,N^2,N} \cdot \left(\frac{\partial}{\partial x} Y_n^{N,N^2,N} \right) \right. \\
& \left. + Y_n^{N,N^2,N} \cdot \left(\Delta W_n^{N^2,N} + \frac{1}{2} \left(\Delta W_n^{N^2,N} \right)^2 - \frac{T}{2N^2} \sum_{j=1}^N \mu_j (g_j)^2 \right) \right) \quad (61)
\end{aligned}$$

for all $n \in \{0, 1, \dots, N^2 - 1\}$ and all $N \in \mathbb{N}$ here.

The linear implicit Euler scheme applied to the SPDE (60) which we denote by $\mathcal{F}/\mathcal{B}(H)$ -measurable mappings $Z_n^N : \Omega \rightarrow H$, $n \in \{0, 1, \dots, N^3\}$,

$N \in \mathbb{N}$, is given by $Z_0^N = P_N(X_0)$ and

$$Z_{n+1}^N = P_N e^{A \frac{T}{N^3}} \left(Z_n^N - \frac{T}{N^3} \cdot Z_n^N \cdot \left(\frac{\partial}{\partial x} Z_n^N \right) + Z_n^N \cdot \Delta W_n^{N^3, N} \right) \quad (62)$$

for all $n \in \{0, 1, \dots, N^3 - 1\}$ and all $N \in \mathbb{N}$ here.

Recently, it has been shown in [28] that Euler's method fails to converge in the root mean square sense to the exact solution of a SODE with a superlinearly growing drift coefficient of the form (60) (see Theorem 2 in [28] for the precise statement of this result) although pathwise convergence often holds (see, e.g., [17, 33] for SODEs and [18, 29, 49] for SPDEs). That is the reason why we consider the pathwise instead of the root mean square approximation error in this section. More precisely, in Figure 9 the pathwise approximation error $\|X_T(\omega) - Z_{N^3}^N(\omega)\|_H^2$ of the linear implicit Euler approximation $Z_{N^3}^N$ (see (62)) and the pathwise approximation error $\|X_T(\omega) - Y_{N^2}^{N, N^2, N}(\omega)\|_H$ of the approximation $Y_{N^2}^{N, N^2, N}$ in this article (see (61)) is plotted against the precise number of independent standard normal random variables needed to compute the corresponding approximation for $N \in \{2, 4, 8, 16, 32, 64, 128\}$ and one random $\omega \in \Omega$.

6 Proof of Theorem 1

The notation

$$\|Z\|_{L^p(\Omega; E)} := \left(\mathbb{E} \|Z\|_E^p \right)^{\frac{1}{p}} \in [0, \infty]$$

is used throughout this section for an \mathbb{R} -Banach space $(E, \|\cdot\|_E)$, a $\mathcal{F}/\mathcal{B}(E)$ -measurable mapping $Z : \Omega \rightarrow E$ and a real number $p \in [1, \infty)$. We also use the following simple lemma (see, e.g., Lemma 1 in [34] and also Theorem 37.5 in [56]).

Lemma 1. *Let Assumptions 1-4 in Section 2 be fulfilled. Then we have*

$$\|(-tA)^r e^{At}\|_{L(H)} \leq 1 \quad \text{and} \quad \|(-tA)^{-r} (e^{At} - I)\|_{L(H)} \leq 1$$

for every $t \in (0, \infty)$ and every $r \in [0, 1]$.

In the following Theorem 1 is established. First of all, note that the exact

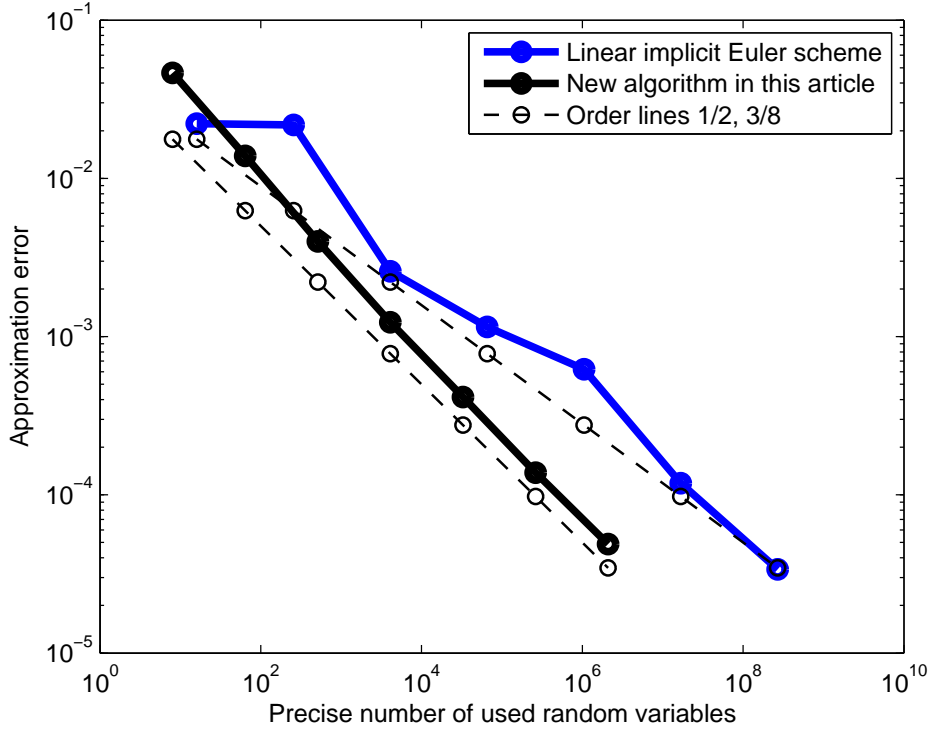


Figure 9: SPDE (60): Pathwise approximation error $\|X_T(\omega) - Z_{N^3}^N(\omega)\|_H^2$ of the linear implicit Euler approximation $Z_{N^3}^N$ (see (62)) and pathwise approximation error $\|X_T(\omega) - Y_{N^2}^{N,N^2,N}(\omega)\|_H$ of the approximation $Y_{N^2}^{N,N^2,N}$ in this article (see (61)) against the precise number of independent standard normal random variables needed to compute the corresponding approximation for $N \in \{2, 4, 8, 16, 32, 64, 128\}$ and one random $\omega \in \Omega$.

solution of the SPDE (23) satisfies

$$\begin{aligned}
X_{mh} &= e^{Amh}\xi + \int_0^{mh} e^{A(mh-s)} F(X_s) ds + \int_0^{mh} e^{A(mh-s)} B(X_s) dW_s \\
&= e^{Amh}\xi + \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(mh-s)} F(X_s) ds + \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(mh-s)} B(X_s) dW_s
\end{aligned} \tag{63}$$

\mathbb{P} -a.s. for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. Here and below h is the time stepsize $h = h_M = \frac{T}{M}$ with $M \in \mathbb{N}$. In particular, (63) shows

$$\begin{aligned}
P_N(X_{mh}) &= e^{Amh} P_N(\xi) + P_N \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(mh-s)} F(X_s) ds \right) \\
&\quad + P_N \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(mh-s)} B(X_s) dW_s \right)
\end{aligned} \tag{64}$$

\mathbb{P} -a.s. for every $m \in \{0, 1, \dots, M\}$ and every $N, M \in \mathbb{N}$. In order to estimate the difference of the exact solution (63) and the numerical solution (27) we rewrite the numerical method (27) in some sense. More precisely, the identity

$$\begin{aligned}
&\frac{1}{2} B'(Y_m^{N,M,K}) \left(B(Y_m^{N,M,K}) \Delta W_m^{M,K} \right) \Delta W_m^{M,K} \\
&\quad - \frac{T}{2M} \sum_{\substack{j \in J_K \\ \mu_j \neq 0}} \mu_j B'(Y_m^{N,M,K}) \left(B(Y_m^{N,M,K}) g_j \right) g_j \\
&= \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} B'(Y_m^{N,M,K}) \left(\int_{\frac{mT}{M}}^s B(Y_m^{N,M,K}) dW_u^K \right) dW_s^K \tag{65}
\end{aligned}$$

\mathbb{P} -a.s. holds for all $m \in \{0, 1, \dots, M-1\}$ and all $N, M, K \in \mathbb{N}$. The proof of (65) can be found in Subsection 6.7. Using (65) shows that the numerical solution (27) fulfills

$$\begin{aligned}
Y_{m+1}^{N,M,K} &= P_N e^{A \frac{T}{M}} \left(Y_m^{N,M,K} + \frac{T}{M} \cdot F(Y_m^{N,M,K}) + \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} B(Y_m^{N,M,K}) dW_s^K \right. \\
&\quad \left. + \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} B'(Y_m^{N,M,K}) \left(\int_{\frac{mT}{M}}^s B(Y_m^{N,M,K}) dW_u^K \right) dW_s^K \right) \tag{66}
\end{aligned}$$

\mathbb{P} -a.s. for every $m \in \{0, 1, \dots, M-1\}$ and every $N, M, K \in \mathbb{N}$. Therefore, the numerical solution (27) satisfies

$$\begin{aligned} Y_m^{N,M,K} &= e^{Amh} P_N(\xi) + P_N \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} F(Y_l^{N,M,K}) ds \right) \\ &+ P_N \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} B(Y_l^{N,M,K}) dW_s^K \right) \\ &+ P_N \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} B'(Y_l^{N,M,K}) \left(\int_{lh}^s B(Y_l^{N,M,K}) dW_u^K \right) dW_s^K \right) \end{aligned} \quad (67)$$

\mathbb{P} -a.s. for every $m \in \{0, 1, \dots, M\}$ and every $N, M, K \in \mathbb{N}$. In order to appraise $\mathbb{E} \|X_{mh} - Y_m^{N,M,K}\|_H^2$ for $m \in \{0, 1, \dots, M\}$ and $N, M, K \in \mathbb{N}$, we define the $\mathcal{F}/\mathcal{B}(H)$ -measurable mappings $Z_m^{N,M,K} : \Omega \rightarrow H$, $m \in \{0, 1, \dots, M\}$, $N, M, K \in \mathbb{N}$, by

$$\begin{aligned} Z_m^{N,M,K} &:= e^{Amh} P_N(\xi) + P_N \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} F(X_{lh}) ds \right) \\ &+ P_N \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} B(X_{lh}) dW_s^K \right) \\ &+ P_N \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} B'(X_{lh}) \left(\int_{lh}^s B(X_{lh}) dW_u^K \right) dW_s^K \right) \end{aligned} \quad (68)$$

\mathbb{P} -a.s. for every $m \in \{0, 1, \dots, M\}$ and every $N, M, K \in \mathbb{N}$. The inequality

$$(a_1 + \dots + a_n)^2 \leq n((a_1)^2 + \dots + (a_n)^2) \quad (69)$$

for every $a_1, \dots, a_n \in \mathbb{R}$ and every $n \in \mathbb{N}$ then shows

$$\begin{aligned} \mathbb{E} \|X_{mh} - Y_m^{N,M,K}\|_H^2 &\leq 3 \cdot \mathbb{E} \|X_{mh} - P_N(X_{mh})\|_H^2 \\ &+ 3 \cdot \mathbb{E} \|P_N(X_{mh}) - Z_m^{N,M,K}\|_H^2 + 3 \cdot \mathbb{E} \|Z_m^{N,M,K} - Y_m^{N,M,K}\|_H^2 \end{aligned} \quad (70)$$

for every $m \in \{0, 1, \dots, M\}$ and every $N, M, K \in \mathbb{N}$. In order to estimate the expressions $\mathbb{E} \|X_{mh} - P_N(X_{mh})\|_H^2$, $\mathbb{E} \|P_N(X_{mh}) - Z_m^{N,M,K}\|_H^2$ and $\mathbb{E} \|Z_m^{N,M,K} - Y_m^{N,M,K}\|_H^2$ for $m \in \{0, 1, \dots, M\}$ and $N, M, K \in \mathbb{N}$, the real number $R \in (0, \infty)$ satisfying

$$\mathbb{E} \|B(X_t)\|_{HS(U_0, V_\delta)}^2 \leq R, \quad \|F'(v)\|_{L(H)} \leq R, \quad \|F''(v)\|_{L^{(2)}(V_\beta, H)} \leq R,$$

$$\begin{aligned}
& \mathbb{E} \|F(X_t)\|_H^2 \leq R, \quad \|B'(v)\|_{L(H, HS(U_0, H))} \leq R, \quad \|B''(v)\|_{L^{(2)}(V_\beta, HS(U_0, H))} \leq R, \\
& \mathbb{E} \|(-A)^\gamma X_t\|_H^2 = \mathbb{E} \|X_t\|_{V_\gamma}^2 \leq R, \quad \mathbb{E} \|X_{t_2} - X_{t_1}\|_{V_\beta}^4 \leq R |t_2 - t_1|^{\min(4(\gamma-\beta), 2)}, \\
& c + \frac{1}{(1-\gamma)} + \frac{1}{(1-2\vartheta)} + \frac{1}{(1-2\delta)} + T + \|A^{-1}\|_{L(H)} \leq R
\end{aligned}$$

for every $v \in V_\beta$ and every $t, t_1, t_2 \in [0, T]$ is used throughout this proof. Due to Assumptions 1-4 in Section 2 and Proposition 1 such a real number indeed exists. For the spatial discretization error $\mathbb{E} \|X_{mh} - P_N(X_{mh})\|_H^2$ we then obtain

$$\begin{aligned}
& \mathbb{E} \|X_{mh} - P_N(X_{mh})\|_H^2 \\
&= \mathbb{E} \|(I - P_N) X_{mh}\|_H^2 = \mathbb{E} \|(-A)^{-\gamma} (I - P_N) (-A)^\gamma X_{mh}\|_H^2 \quad (71) \\
&\leq \|(-A)^{-\gamma} (I - P_N)\|_{L(H)}^2 \mathbb{E} \|X_{mh}\|_{V_\gamma}^2 \leq R (r_N)^2
\end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $N, M \in \mathbb{N}$ where here and below the real numbers $(r_N)_{N \in \mathbb{N}} \subset \mathbb{R}$ are given by

$$r_N := \|(-A)^{-\gamma} (I - P_N)\|_{L(H)} = \left(\inf_{i \in \mathcal{I} \setminus \mathcal{I}_N} \lambda_i \right)^{-\gamma}$$

for every $N \in \mathbb{N}$. The rest of this proof is then divided into six parts. In the first part (see Subsection 6.1) we establish

$$\mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} F(X_s) - e^{A(m-l)h} F(X_{lh})) ds \right\|_H^2 \leq \frac{36R^8}{M^{\min(4(\gamma-\beta), 2\gamma)}} \quad (72)$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. We show

$$\mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(mh-s)} B(X_s) d(W_s - W_s^K) \right\|_H^2 \leq 4R^5 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \quad (73)$$

for every $m \in \{0, 1, \dots, M\}$ and every $M, K \in \mathbb{N}$ in the second part (see Subsection 6.2) and

$$\mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} - e^{A(m-l)h}) B(X_s) dW_s^K \right\|_H^2 \leq \frac{3R^4}{M^{(1+2\delta)}} \quad (74)$$

for every $m \in \{0, 1, \dots, M\}$ and every $M, K \in \mathbb{N}$ in the third part (see Subsection 6.3). The fourth part (see Subsection 6.4) gives

$$\mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} \int_0^1 B''(X_{lh} + r(X_s - X_{lh}))(X_s - X_{lh}, X_s - X_{lh}) \cdot (1-r) dr dW_s^K \right\|_H^2 \leq \frac{R^6}{M^{\min(4(\gamma-\beta), 2)}} \quad (75)$$

and in the fifth part (see Subsection 6.5) we obtain

$$\begin{aligned} \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} B'(X_{lh}) \left(X_s - X_{lh} - \int_{lh}^s B(X_{lh}) dW_u^K \right) dW_s^K \right\|_H^2 \\ \leq \frac{20R^{13}}{M^{\min(4(\gamma-\beta), 2\gamma)}} + 20R^{11} \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \end{aligned} \quad (76)$$

for every $m \in \{0, 1, \dots, M\}$ and every $M, K \in \mathbb{N}$. The inequalities (72)-(76) are used below to estimate $\mathbb{E} \|P_N(X_{mh}) - Z_m^{N,M,K}\|_H^2$ for $m \in \{0, 1, \dots, M\}$ and $N, M, K \in \mathbb{N}$ in (70). In the sixth part (see Subsection 6.6) we estimate

$$\mathbb{E} \|Z_m^{N,M,K} - Y_m^{N,M,K}\|_H^2 \leq \frac{9R^4}{M} \left(\sum_{l=0}^{m-1} \mathbb{E} \|X_{lh} - Y_l^{N,M,K}\|_H^2 \right) \quad (77)$$

for every $m \in \{0, 1, \dots, M\}$ and every $N, M, K \in \mathbb{N}$ by using the global Lipschitz continuity of the coefficients $F : V_\beta \rightarrow H$ (see Assumption 2) and $B : V_\beta \rightarrow HS(U_0, H)$ (see Assumption 3). Combining (70), (71) and (77) then yields

$$\begin{aligned} \mathbb{E} \|X_{mh} - Y_m^{N,M,K}\|_H^2 &\leq 3R(r_N)^2 \\ &+ 3 \cdot \mathbb{E} \|P_N(X_{mh}) - Z_m^{N,M,K}\|_H^2 + \frac{27R^4}{M} \left(\sum_{l=0}^{m-1} \mathbb{E} \|X_{lh} - Y_l^{N,M,K}\|_H^2 \right) \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $N, M, K \in \mathbb{N}$. Hence, (64), (68) and

(69) show

$$\begin{aligned}
& \mathbb{E} \left\| X_{mh} - Y_m^{N,M,K} \right\|_H^2 \\
& \leq 9 \cdot \mathbb{E} \left\| P_N \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} F(X_s) - e^{A(m-l)h} F(X_{lh})) ds \right) \right\|_H^2 \\
& + 9 \cdot \mathbb{E} \left\| P_N \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(mh-s)} B(X_s) d(W_s - W_s^K) \right) \right\|_H^2 \\
& + 9 \cdot \mathbb{E} \left\| P_N \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} B(X_s) - e^{A(m-l)h} B(X_{lh})) dW_s^K \right. \right. \\
& \quad \left. \left. - \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} B'(X_{lh}) \left(\int_{lh}^s B(X_{lh}) dW_u^K \right) dW_s^K \right) \right\|_H^2 \\
& + 3R(r_N)^2 + \frac{27R^4}{M} \left(\sum_{l=0}^{m-1} \mathbb{E} \left\| X_{lh} - Y_l^{N,M,K} \right\|_H^2 \right)
\end{aligned}$$

and using $\|P_N(v)\|_H \leq \|v\|_H$ for all $v \in H$ additionally gives

$$\begin{aligned}
& \mathbb{E} \left\| X_{mh} - Y_m^{N,M,K} \right\|_H^2 \\
& \leq 9 \cdot \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} F(X_s) - e^{A(m-l)h} F(X_{lh})) ds \right\|_H^2 \\
& + 9 \cdot \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(mh-s)} B(X_s) d(W_s - W_s^K) \right\|_H^2 \\
& + 9 \cdot \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} B(X_s) - e^{A(m-l)h} B(X_{lh})) dW_s^K \right. \\
& \quad \left. - \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} B'(X_{lh}) \left(\int_{lh}^s B(X_{lh}) dW_u^K \right) dW_s^K \right\|_H^2 \\
& + 3R(r_N)^2 + \frac{27R^4}{M} \left(\sum_{l=0}^{m-1} \mathbb{E} \left\| X_{lh} - Y_l^{N,M,K} \right\|_H^2 \right)
\end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $N, M, K \in \mathbb{N}$. Therefore, (72) and

(73) yield

$$\begin{aligned}
\mathbb{E} \|X_{mh} - Y_m^{N,M,K}\|_H^2 &\leq \frac{324R^8}{M^{\min(4(\gamma-\beta), 2\gamma)}} + 36R^5 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \\
&+ 18 \cdot \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} - e^{A(m-l)h}) B(X_s) dW_s^K \right\|_H^2 \\
&+ 18 \cdot \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} (B(X_s) - B(X_{lh})) dW_s^K \right. \\
&\quad \left. - \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} B'(X_{lh}) \left(\int_{lh}^s B(X_{lh}) dW_u^K \right) dW_s^K \right\|_H^2 \\
&+ 3R(r_N)^2 + \frac{27R^4}{M} \left(\sum_{l=0}^{m-1} \mathbb{E} \|X_{lh} - Y_l^{N,M,K}\|_H^2 \right)
\end{aligned}$$

and (74) shows

$$\begin{aligned}
\mathbb{E} \|X_{mh} - Y_m^{N,M,K}\|_H^2 &\leq \frac{324R^8}{M^{\min(4(\gamma-\beta), 2\gamma)}} + 36R^5 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} + \frac{54R^4}{M(1+2\delta)} \\
&+ 18 \cdot \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} (B(X_s) - B(X_{lh})) dW_s^K \right. \\
&\quad \left. - \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} B'(X_{lh}) \left(\int_{lh}^s B(X_{lh}) dW_u^K \right) dW_s^K \right\|_H^2 \\
&+ 3R(r_N)^2 + \frac{27R^4}{M} \left(\sum_{l=0}^{m-1} \mathbb{E} \|X_{lh} - Y_l^{N,M,K}\|_H^2 \right)
\end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $N, M, K \in \mathbb{N}$. The fact

$$\begin{aligned}
B(X_s) - B(X_{lh}) &= B'(X_{lh})(X_s - X_{lh}) \\
&+ \int_0^1 B''(X_{lh} + r(X_s - X_{lh}))(X_s - X_{lh}, X_s - X_{lh})(1-r) dr
\end{aligned}$$

for every $s \in [lh, (l+1)h]$, $l \in \{0, 1, \dots, M-1\}$ and every $M \in \mathbb{N}$ then yields

$$\begin{aligned}
\mathbb{E} \|X_{mh} - Y_m^{N,M,K}\|_H^2 &\leq \frac{324R^8}{M^{\min(4(\gamma-\beta), 2\gamma)}} + 36R^5 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} + \frac{54R^4}{M^{(1+2\delta)}} \\
&+ 36 \cdot \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} B'(X_{lh}) \left(X_s - X_{lh} - \int_{lh}^s B(X_{lh}) dW_u^K \right) dW_s^K \right\|_H^2 \\
&+ 36 \cdot \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} \int_0^1 B''(X_{lh} + r(X_s - X_{lh})) (X_s - X_{lh}, \right. \\
&\quad \left. X_s - X_{lh}) (1-r) dr dW_s^K \right\|_H^2 \\
&+ 3R(r_N)^2 + \frac{27R^4}{M} \left(\sum_{l=0}^{m-1} \mathbb{E} \|X_{lh} - Y_l^{N,M,K}\|_H^2 \right)
\end{aligned}$$

for every $m \in \{0, 1, \dots, M-1\}$ and every $N, M, K \in \mathbb{N}$. Therefore, (75) and (76) give

$$\begin{aligned}
\mathbb{E} \|X_{mh} - Y_m^{N,M,K}\|_H^2 &\leq \frac{324R^8}{M^{\min(4(\gamma-\beta), 2\gamma)}} + 756R^{11} \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} + \frac{54R^4}{M^{(1+2\delta)}} \\
&+ \frac{720R^{13}}{M^{\min(4(\gamma-\beta), 2\gamma)}} + \frac{36R^6}{M^{\min(4(\gamma-\beta), 2)}} \\
&+ 3R(r_N)^2 + \frac{27R^4}{M} \left(\sum_{l=0}^{m-1} \mathbb{E} \|X_{lh} - Y_l^{N,M,K}\|_H^2 \right)
\end{aligned}$$

and hence

$$\begin{aligned}
\mathbb{E} \|X_{mh} - Y_m^{N,M,K}\|_H^2 &\leq (324R^8 + 54R^4 + 720R^{13} + 36R^6) \frac{1}{M^{\min(4(\gamma-\beta), 2\gamma)}} \\
&+ 756R^{11} \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} + 3R(r_N)^2 + \frac{27R^4}{M} \left(\sum_{l=0}^{m-1} \mathbb{E} \|X_{lh} - Y_l^{N,M,K}\|_H^2 \right)
\end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $N, M, K \in \mathbb{N}$. Gronwall's lemma thus shows

$$\begin{aligned}
\mathbb{E} \|X_{mh} - Y_m^{N,M,K}\|_H^2 &\leq e^{27R^4} \left(\frac{(324R^8 + 54R^4 + 720R^{13} + 36R^6)}{M^{\min(4(\gamma-\beta), 2\gamma)}} \right. \\
&\quad \left. + 756R^{11} \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} + 3R(r_N)^2 \right)
\end{aligned}$$

and hence

$$\begin{aligned} \mathbb{E} \|X_{mh} - Y_m^{N,M,K}\|_H^2 \\ \leq 1134R^{13}e^{27R^4} \left((r_N)^2 + \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} + M^{-\min(4(\gamma-\beta), 2\gamma)} \right) \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $N, M, K \in \mathbb{N}$. Finally, we obtain

$$\begin{aligned} & \left(\mathbb{E} \|X_{mh} - Y_m^{N,M,K}\|_H^2 \right)^{\frac{1}{2}} \\ & \leq 34R^7 e^{14R^4} \left(\left(\inf_{i \in \mathcal{I} \setminus \mathcal{I}_N} \lambda_i \right)^{-\gamma} + \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^\alpha + M^{-\min(2(\gamma-\beta), \gamma)} \right) \\ & \leq e^{20R^4} \left(\left(\inf_{i \in \mathcal{I} \setminus \mathcal{I}_N} \lambda_i \right)^{-\gamma} + \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^\alpha + M^{-\min(2(\gamma-\beta), \gamma)} \right) \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $N, M, K \in \mathbb{N}$.

6.1 Temporal discretization error: Proof of (72)

First of all, we have

$$\begin{aligned} & \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} F(X_s) - e^{A(m-l)h} F(X_{lh})) ds \right\|_{L^2(\Omega; H)} \\ & \leq \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \| (e^{A(mh-s)} - e^{A(m-l)h}) F(X_s) \|_{L^2(\Omega; H)} ds \\ & \quad + \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} (F(X_s) - F(X_{lh})) ds \right\|_{L^2(\Omega; H)} \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. Using

$$\begin{aligned} F(X_s) - F(X_{lh}) &= F'(X_{lh}) (X_s - X_{lh}) \\ &+ \int_0^1 F''(X_{lh} + r(X_s - X_{lh})) (X_s - X_{lh}, X_s - X_{lh}) (1-r) dr \end{aligned}$$

for all $s \in [lh, (l+1)h]$, $l \in \{0, 1, \dots, M-1\}$ and all $M \in \mathbb{N}$ then shows

$$\begin{aligned}
& \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} F(X_s) - e^{A(m-l)h} F(X_{lh})) ds \right\|_{L^2(\Omega; H)} \\
& \leq \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \|e^{A(mh-s)} - e^{A(m-l)h}\|_{L(H)} \|F(X_s)\|_{L^2(\Omega; H)} ds \\
& \quad + \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} F'(X_{lh}) (X_s - X_{lh}) ds \right\|_{L^2(\Omega; H)} \\
& \quad + \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \int_0^1 \|F''(X_{lh} + r(X_s - X_{lh})) (X_s - X_{lh}, X_s - X_{lh})\|_{L^2(\Omega; H)} \\
& \quad (1-r) dr ds
\end{aligned}$$

and hence

$$\begin{aligned}
& \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} F(X_s) - e^{A(m-l)h} F(X_{lh})) ds \right\|_{L^2(\Omega; H)} \\
& \leq R \left(2h + \sum_{l=0}^{m-2} \int_{lh}^{(l+1)h} \|e^{A(mh-s)} - e^{A(m-l)h}\|_{L(H)} ds \right) \\
& \quad + \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} F'(X_{lh}) (X_s - X_{lh}) ds \right\|_{L^2(\Omega; H)} \\
& \quad + \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \int_0^1 \|R \|X_s - X_{lh}\|_{V_\beta}^2\|_{L^2(\Omega; \mathbb{R})} (1-r) dr ds
\end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned}
& \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} F(X_s) - e^{A(m-l)h} F(X_{lh})) ds \right\|_{L^2(\Omega; H)} \\
& \leq R \left(2h + \sum_{l=0}^{m-2} \int_{lh}^{(l+1)h} \|Ae^{A(mh-s)}\|_{L(H)} \|A^{-1} (e^{A(s-lh)} - I)\|_{L(H)} ds \right) \\
& \quad + \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} F'(X_{lh}) (X_s - X_{lh}) ds \right\|_{L^2(\Omega; H)} \\
& \quad + \frac{R}{2} \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (\mathbb{E} \|X_s - X_{lh}\|_{V_\beta}^4)^{\frac{1}{2}} ds \right)
\end{aligned}$$

and Lemma 1 gives

$$\begin{aligned}
& \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} F(X_s) - e^{A(m-l)h} F(X_{lh})) ds \right\|_{L^2(\Omega; H)} \\
& \leq R \left(2h + \sum_{l=0}^{m-2} \int_{lh}^{(l+1)h} \frac{(s-lh)}{(mh-s)} ds \right) \\
& \quad + \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \| e^{A(m-l)h} F'(X_{lh}) ((e^{A(s-lh)} - I) X_{lh}) \|_{L^2(\Omega; H)} ds \\
& \quad + \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \left\| e^{A(m-l)h} F'(X_{lh}) \left(\int_{lh}^s e^{A(s-u)} F(X_u) du \right) \right\|_{L^2(\Omega; H)} ds \\
& \quad + \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} F'(X_{lh}) \left(\int_{lh}^s e^{A(s-u)} B(X_u) dW_u \right) ds \right\|_{L^2(\Omega; H)} \\
& \quad + \frac{R}{2} \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \left(R(s-lh)^{\min(4(\gamma-\beta), 2)} \right)^{\frac{1}{2}} ds \right)
\end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. This shows

$$\begin{aligned}
& \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} F(X_s) - e^{A(m-l)h} F(X_{lh})) ds \right\|_{L^2(\Omega; H)} \\
& \leq R \left(2h + \sum_{l=0}^{m-2} \int_{lh}^{(l+1)h} \frac{(s-lh)}{(m-l-1)h} ds \right) \\
& \quad + \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \| F'(X_{lh}) ((e^{A(s-lh)} - I) X_{lh}) \|_{L^2(\Omega; H)} ds \\
& \quad + \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \left\| F'(X_{lh}) \left(\int_{lh}^s e^{A(s-u)} F(X_u) du \right) \right\|_{L^2(\Omega; H)} ds \\
& \quad + \left\{ \sum_{l=0}^{m-1} \mathbb{E} \left\| \int_{lh}^{(l+1)h} e^{A(m-l)h} F'(X_{lh}) \left(\int_{lh}^s e^{A(s-u)} B(X_u) dW_u \right) ds \right\|_H^2 \right\}^{\frac{1}{2}} \\
& \quad + \frac{R^2}{2} \left(\sum_{l=0}^{m-1} h^{(1+\min(2(\gamma-\beta), 1))} \right)
\end{aligned}$$

and

$$\begin{aligned}
& \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} F(X_s) - e^{A(m-l)h} F(X_{lh})) ds \right\|_{L^2(\Omega; H)} \\
& \leq R \left(2h + \sum_{l=0}^{m-2} \frac{h}{2(m-l-1)} \right) + \frac{1}{2} R^2 T h^{\min(2(\gamma-\beta), 1)} \\
& \quad + R \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \| (e^{A(s-lh)} - I) X_{lh} \|_{L^2(\Omega; H)} ds \right) \\
& \quad + R \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \left\| \int_{lh}^s e^{A(s-u)} F(X_u) du \right\|_{L^2(\Omega; H)} ds \right) \\
& \quad + \sqrt{h} \left\{ \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \mathbb{E} \left\| F'(X_{lh}) \left(\int_{lh}^s e^{A(s-u)} B(X_u) dW_u \right) \right\|_H^2 ds \right\}^{\frac{1}{2}}
\end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. Hence, we have

$$\begin{aligned}
& \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} F(X_s) - e^{A(m-l)h} F(X_{lh})) ds \right\|_{L^2(\Omega; H)} \\
& \leq R \left(2h + \frac{h}{2} \left(\sum_{l=1}^{m-1} \frac{1}{l} \right) \right) + \frac{1}{2} R^3 h^{\min(2(\gamma-\beta), 1)} \\
& \quad + R \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \| (-A)^{-\gamma} (e^{A(s-lh)} - I) \|_{L(H)} \| (-A)^{\gamma} X_{lh} \|_{L^2(\Omega; H)} ds \right) \\
& \quad + R \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \int_{lh}^s \| F(X_u) \|_{L^2(\Omega; H)} du ds \right) \\
& \quad + R \sqrt{h} \left\{ \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \mathbb{E} \left\| \int_{lh}^s e^{A(s-u)} B(X_u) dW_u \right\|_H^2 ds \right\}^{\frac{1}{2}}
\end{aligned}$$

and Lemma 1 shows

$$\begin{aligned}
& \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} F(X_s) - e^{A(m-l)h} F(X_{lh})) ds \right\|_{L^2(\Omega; H)} \\
& \leq R \left(2h + \frac{h}{2} (1 + \ln(M)) \right) + \frac{1}{2} R^4 M^{-\min(2(\gamma-\beta), 1)} \\
& \quad + R^2 \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (s - lh)^\gamma ds \right) + \frac{1}{2} R^2 M h^2 \\
& \quad + R\sqrt{h} \left\{ \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \int_{lh}^s \mathbb{E} \|e^{A(s-u)} B(X_u)\|_{HS(U_0, H)}^2 du ds \right\}^{\frac{1}{2}}
\end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned}
& \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} F(X_s) - e^{A(m-l)h} F(X_{lh})) ds \right\|_{L^2(\Omega; H)} \\
& \leq Rh \left(\frac{5}{2} + \frac{1}{2} \ln(M) \right) + \frac{1}{2} R^4 M^{-\min(2(\gamma-\beta), 1)} + R^2 M h^{(1+\gamma)} + \frac{1}{2} R^3 h \\
& \quad + R\sqrt{h} \left\{ \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \int_{lh}^s \mathbb{E} \|B(X_u)\|_{HS(U_0, H)}^2 du ds \right\}^{\frac{1}{2}}
\end{aligned}$$

and hence

$$\begin{aligned}
& \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} F(X_s) - e^{A(m-l)h} F(X_{lh})) ds \right\|_{L^2(\Omega; H)} \\
& \leq R^2 M^{-1} \left(\frac{5}{2} + \frac{1}{2} \ln(M) \right) + \frac{1}{2} R^4 M^{-\min(2(\gamma-\beta), 1)} + R^4 M^{-\gamma} + \frac{1}{2} R^4 M^{-1} \\
& \quad + R\sqrt{h} \left\{ \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \int_{lh}^s \|(-A)^{-\delta}\|_{L(H)}^2 \mathbb{E} \|(-A)^\delta B(X_u)\|_{HS(U_0, H)}^2 du ds \right\}^{\frac{1}{2}}
\end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. This yields

$$\begin{aligned}
& \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} F(X_s) - e^{A(m-l)h} F(X_{lh})) ds \right\|_{L^2(\Omega; H)} \\
& \leq \frac{5}{2} R^2 M^{-1} (1 + \ln(M)) + \frac{1}{2} R^4 M^{-\min(2(\gamma-\beta), 1)} + R^4 M^{-\gamma} + \frac{1}{2} R^4 M^{-1} \\
& \quad + R^2 \sqrt{h} \left(\frac{1}{2} M h^2 \right)^{\frac{1}{2}}
\end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. The estimate

$$\begin{aligned} 1 + \ln(x) &= 1 + \int_1^x \frac{1}{s} ds \leq 1 + \int_1^x \frac{1}{s^{(1-r)}} ds = 1 + \left[\frac{s^r}{r} \right]_{s=1}^{s=x} \\ &= 1 + \frac{(x^r - 1)}{r} = \frac{x^r}{r} - \frac{(1-r)}{r} \leq \frac{x^r}{r} \end{aligned}$$

for every $r \in (0, 1]$ and every $x \in [1, \infty)$ then shows

$$\begin{aligned} & \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} F(X_s) - e^{A(m-l)h} F(X_{lh})) ds \right\|_{L^2(\Omega; H)} \\ & \leq \frac{5}{2} R^2 \frac{M^{(1-\gamma)}}{M(1-\gamma)} + \frac{R^4}{2M^{\min(2(\gamma-\beta), 1)}} + \frac{R^4}{M^\gamma} + \frac{R^4}{2M} + R^2 \sqrt{h} (Th)^{\frac{1}{2}} \\ & \leq \frac{5R^4}{2M^\gamma} + \frac{R^4}{2M^{\min(2(\gamma-\beta), 1)}} + \frac{R^4}{M^\gamma} + \frac{R^4}{2M} + \frac{R^4}{M} \end{aligned}$$

and finally

$$\begin{aligned} & \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} F(X_s) - e^{A(m-l)h} F(X_{lh})) ds \right\|_{L^2(\Omega; H)} \\ & \leq \left(\frac{5}{2} + \frac{1}{2} + 1 + \frac{1}{2} + 1 \right) \frac{R^4}{M^{\min(2(\gamma-\beta), \gamma)}} \leq \frac{6R^4}{M^{\min(2(\gamma-\beta), \gamma)}} \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$.

6.2 Noise discretization error: Proof of (73)

In this subsection we have

$$\begin{aligned} & \mathbb{E} \left\| \int_s^t e^{A(t-u)} B(X_u) d(W_u - W_u^K) \right\|_H^2 \\ &= \mathbb{E} \left\| \sum_{\substack{j \in \mathcal{J} \setminus \mathcal{J}_K \\ \mu_j \neq 0}} \int_s^t e^{A(t-u)} B(X_u) g_j d\langle g_j, W_u \rangle_U \right\|_H^2 \\ &= \sum_{\substack{j \in \mathcal{J} \setminus \mathcal{J}_K \\ \mu_j \neq 0}} \mu_j \left(\int_s^t \mathbb{E} \|e^{A(t-u)} B(X_u) g_j\|_H^2 du \right) \end{aligned}$$

and hence

$$\begin{aligned}
& \mathbb{E} \left\| \int_s^t e^{A(t-u)} B(X_u) d(W_u - W_u^K) \right\|_H^2 \\
&= \sum_{\substack{j \in \mathcal{J} \setminus \mathcal{J}_K \\ \mu_j \neq 0}} \mu_j \left(\int_s^t \mathbb{E} \|e^{A(t-u)} B(X_u) Q^{-\alpha} (Q^\alpha g_j)\|_H^2 du \right) \\
&= \sum_{\substack{j \in \mathcal{J} \setminus \mathcal{J}_K \\ \mu_j \neq 0}} (\mu_j)^{(1+2\alpha)} \left(\int_s^t \mathbb{E} \|e^{A(t-u)} B(X_u) Q^{-\alpha} g_j\|_H^2 du \right)
\end{aligned}$$

for every $s, t \in [0, T]$ with $s \leq t$ and every $K \in \mathbb{N}$. This shows

$$\begin{aligned}
& \mathbb{E} \left\| \int_s^t e^{A(t-u)} B(X_u) d(W_u - W_u^K) \right\|_H^2 \\
&\leq \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \left(\sum_{\substack{j \in \mathcal{J} \setminus \mathcal{J}_K \\ \mu_j \neq 0}} \mu_j \left(\int_s^t \mathbb{E} \|e^{A(t-u)} B(X_u) Q^{-\alpha} g_j\|_H^2 du \right) \right) \\
&\leq \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \left(\sum_{j \in \mathcal{J}} \mu_j \int_s^t \mathbb{E} \|e^{A(t-u)} B(X_u) Q^{-\alpha} g_j\|_H^2 du \right)
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E} \left\| \int_s^t e^{A(t-u)} B(X_u) d(W_u - W_u^K) \right\|_H^2 \\
&\leq \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \left(\int_s^t \mathbb{E} \|e^{A(t-u)} B(X_u) Q^{-\alpha}\|_{HS(U_0, H)}^2 du \right) \\
&\leq \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \left(\int_s^t (t-u)^{-2\vartheta} \mathbb{E} \|(-A)^{-\vartheta} B(X_u) Q^{-\alpha}\|_{HS(U_0, H)}^2 du \right)
\end{aligned}$$

for every $s, t \in [0, T]$ with $s \leq t$ and every $K \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned}
& \mathbb{E} \left\| \int_s^t e^{A(t-u)} B(X_u) d(W_u - W_u^K) \right\|_H^2 \\
&\leq c^2 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \left(\int_s^t (t-u)^{-2\vartheta} \mathbb{E} \left[\left(1 + \|X_u\|_{V_\gamma} \right)^2 \right] du \right) \\
&\leq 2c^2 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \left(\int_s^t (t-u)^{-2\vartheta} \left(1 + \mathbb{E} \|X_u\|_{V_\gamma}^2 \right) du \right)
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E} \left\| \int_s^t e^{A(t-u)} B(X_u) d(W_u - W_u^K) \right\|_H^2 \\
& \leq 4R^3 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \left(\int_s^t (t-u)^{-2\vartheta} du \right) \\
& = 4R^3 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \left(\int_0^{(t-s)} u^{-2\vartheta} du \right)
\end{aligned}$$

for every $s, t \in [0, T]$ with $s \leq t$ and every $K \in \mathbb{N}$. Hence, we have

$$\begin{aligned}
& \mathbb{E} \left\| \int_s^t e^{A(t-u)} B(X_u) d(W_u - W_u^K) \right\|_H^2 \\
& \leq 4R^3 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \left[\frac{u^{(1-2\vartheta)}}{(1-2\vartheta)} \right]_{u=0}^{u=(t-s)} \\
& \leq 4R^4 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} (t-s)^{(1-2\vartheta)}
\end{aligned}$$

and finally

$$\mathbb{E} \left\| \int_s^t e^{A(t-u)} B(X_u) d(W_u - W_u^K) \right\|_H^2 \leq 4R^5 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \quad (78)$$

for every $s, t \in [0, T]$ with $s \leq t$ and every $K \in \mathbb{N}$. In particular, we obtain

$$\begin{aligned}
& \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(mh-s)} B(X_s) d(W_s - W_s^K) \right\|_H^2 \\
& = \mathbb{E} \left\| \int_0^{mh} e^{A(mh-s)} B(X_s) d(W_s - W_s^K) \right\|_H^2 \leq 4R^5 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha}
\end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M, K \in \mathbb{N}$ which shows (73).

6.3 Temporal discretization error: Proof of (74)

Here we have

$$\begin{aligned}
& \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} - e^{A(m-l)h}) B(X_s) dW_s^K \right\|_H^2 \\
& \leq \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \mathbb{E} \left\| (e^{A(mh-s)} - e^{A(m-l)h}) B(X_s) \right\|_{HS(U_0, H)}^2 ds \\
& \leq \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \left\| (-A)^{-\delta} (e^{A(mh-s)} - e^{A(m-l)h}) \right\|_{L(H)}^2 \mathbb{E} \left\| (-A)^{\delta} B(X_s) \right\|_{HS(U_0, H)}^2 ds
\end{aligned}$$

and hence

$$\begin{aligned}
& \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} - e^{A(m-l)h}) B(X_s) dW_s^K \right\|_H^2 \\
& \leq R \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \left\| (-A)^{-\delta} (e^{A(mh-s)} - e^{A(m-l)h}) \right\|_{L(H)}^2 ds \right) \\
& \leq R \int_{(m-1)h}^{mh} \left\| (-A)^{-\delta} (e^{A(mh-s)} - e^{Ah}) \right\|_{L(H)}^2 ds \\
& \quad + R \left(\sum_{l=0}^{m-2} \int_{lh}^{(l+1)h} \left\| (-A)^{-1} (e^{A(s-lh)} - I) \right\|_{L(H)}^2 \left\| (-A)^{(1-\delta)} e^{A(mh-s)} \right\|_{L(H)}^2 ds \right)
\end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M, K \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned}
& \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} - e^{A(m-l)h}) B(X_s) dW_s^K \right\|_H^2 \\
& \leq R \int_{(m-1)h}^{mh} \left\| (-A)^{-\delta} (e^{A(s-(m-1)h)} - I) \right\|_{L(H)}^2 ds \\
& \quad + Rh^2 \left(\sum_{l=0}^{m-2} \int_{lh}^{(l+1)h} \left\| (-A)^{(1-\delta)} e^{A(mh-s)} \right\|_{L(H)}^2 ds \right)
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} - e^{A(m-l)h}) B(X_s) dW_s^K \right\|_H^2 \\
& \leq R \int_{(m-1)h}^{mh} (s - (m-1)h)^{2\delta} ds + Rh^2 \left(\sum_{l=0}^{m-2} \int_{lh}^{(l+1)h} (mh-s)^{2(\delta-1)} ds \right) \\
& \leq Rh^{(1+2\delta)} + Rh^3 \left(\sum_{l=0}^{m-2} (m-l-1)^{2(\delta-1)} h^{2(\delta-1)} \right)
\end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M, K \in \mathbb{N}$. This shows

$$\begin{aligned}
& \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} - e^{A(m-l)h}) B(X_s) dW_s^K \right\|_H^2 \\
& \leq Rh^{(1+2\delta)} + Rh^{(1+2\delta)} \left(\sum_{l=1}^{m-1} l^{2(\delta-1)} \right) \leq Rh^{(1+2\delta)} \left(2 + \sum_{l=2}^{\infty} l^{2(\delta-1)} \right) \\
& \leq Rh^{(1+2\delta)} \left(2 + \int_1^{\infty} s^{2(\delta-1)} ds \right)
\end{aligned}$$

and finally

$$\begin{aligned}
& \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (e^{A(mh-s)} - e^{A(m-l)h}) B(X_s) dW_s^K \right\|_H^2 \\
& \leq Rh^{(1+2\delta)} \left(2 + \left[\frac{s^{2(\delta-1)}}{(2\delta-1)} \right]_{s=1}^{s=\infty} \right) = Rh^{(1+2\delta)} \left(2 + \frac{1}{(1-2\delta)} \right) \\
& \leq 3R^2 h^{(1+2\delta)} \leq \frac{3R^4}{M^{(1+2\delta)}}
\end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M, K \in \mathbb{N}$.

6.4 Temporal discretization error: Proof of (75)

We have

$$\begin{aligned} & \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} \int_0^1 B''(X_{lh} + r(X_s - X_{lh}))(X_s - X_{lh}, X_s - X_{lh}) \right. \\ & \quad \left. \cdot (1-r) dr dW_s^K \right\|_H^2 \\ & \leq \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \int_0^1 \mathbb{E} \|B''(X_{lh} + r(X_s - X_{lh}))(X_s - X_{lh}, X_s - X_{lh})\|_{HS(U_0, H)}^2 dr ds \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M, K \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned} & \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} \int_0^1 B''(X_{lh} + r(X_s - X_{lh}))(X_s - X_{lh}, X_s - X_{lh}) \right. \\ & \quad \left. \cdot (1-r) dr dW_s^K \right\|_H^2 \\ & \leq \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \mathbb{E} \left[\left(R \|X_s - X_{lh}\|_{V_\beta}^2 \right)^2 \right] ds = R^2 \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \mathbb{E} \|X_s - X_{lh}\|_{V_\beta}^4 ds \right) \end{aligned}$$

and hence

$$\begin{aligned} & \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} \int_0^1 B''(X_{lh} + r(X_s - X_{lh}))(X_s - X_{lh}, X_s - X_{lh}) \right. \\ & \quad \left. \cdot (1-r) dr dW_s^K \right\|_H^2 \\ & \leq R^2 \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} R(s-lh)^{\min(4(\gamma-\beta), 2)} ds \right) \leq R^3 \left(\sum_{l=0}^{m-1} h^{(1+\min(4(\gamma-\beta), 2))} \right) \\ & \leq R^3 M h^{(1+\min(4(\gamma-\beta), 2))} = R^3 T h^{\min(4(\gamma-\beta), 2)} \leq \frac{R^6}{M^{\min(4(\gamma-\beta), 2)}} \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M, K \in \mathbb{N}$.

6.5 Temporal discretization error: Proof of (76)

In order to show (76), we first estimate

$$\mathbb{E} \left\| X_t - X_s - \int_s^t B(X_s) dW_u^K \right\|_H^2$$

for all $s, t \in [0, T]$ with $s \leq t$ and all $K \in \mathbb{N}$. More precisely, we have

$$\begin{aligned} & \mathbb{E} \left\| X_t - X_s - \int_s^t B(X_s) dW_u^K \right\|_H^2 \leq 5 \cdot \mathbb{E} \left\| (e^{A(t-s)} - I) X_s \right\|_H^2 \\ & + 5 \cdot \mathbb{E} \left\| \int_s^t e^{A(t-u)} F(X_u) du \right\|_H^2 + 5 \cdot \mathbb{E} \left\| \int_s^t e^{A(t-u)} B(X_u) d(W_u - W_u^K) \right\|_H^2 \\ & + 5 \cdot \mathbb{E} \left\| \int_s^t (e^{A(t-u)} - I) B(X_u) dW_u^K \right\|_H^2 + 5 \cdot \mathbb{E} \left\| \int_s^t (B(X_u) - B(X_s)) dW_u^K \right\|_H^2 \end{aligned}$$

and using (78) shows

$$\begin{aligned} & \mathbb{E} \left\| X_t - X_s - \int_s^t B(X_s) dW_u^K \right\|_H^2 \leq 5 \left\| (-A)^{-\gamma} (e^{A(t-s)} - I) \right\|_{L(H)}^2 \mathbb{E} \left\| (-A)^\gamma X_s \right\|_H^2 \\ & + 5(t-s) \left(\int_s^t \mathbb{E} \left\| e^{A(t-u)} F(X_u) \right\|_H^2 du \right) + 20R^5 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \\ & + 5 \left(\int_s^t \mathbb{E} \left\| (e^{A(t-u)} - I) B(X_u) \right\|_{HS(U_0, H)}^2 du \right) \\ & + 5 \left(\int_s^t \mathbb{E} \left\| B(X_u) - B(X_s) \right\|_{HS(U_0, H)}^2 du \right) \end{aligned}$$

for every $s, t \in [0, T]$ with $s \leq t$ and every $K \in \mathbb{N}$. This shows

$$\begin{aligned} & \mathbb{E} \left\| X_t - X_s - \int_s^t B(X_s) dW_u^K \right\|_H^2 \\ & \leq 5R(t-s)^{2\gamma} + 5(t-s) \left(\int_s^t \mathbb{E} \left\| F(X_u) \right\|_H^2 du \right) + 20R^5 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \\ & + 5 \left(\int_s^t \left\| (-A)^{-\delta} (e^{A(t-u)} - I) \right\|_{L(H)}^2 \mathbb{E} \left\| (-A)^\delta B(X_u) \right\|_{HS(U_0, H)}^2 du \right) \\ & + 5R^2 \left(\int_s^t \mathbb{E} \left\| X_u - X_s \right\|_H^2 du \right) \end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E} \left\| X_t - X_s - \int_s^t B(X_s) dW_u^K \right\|_H^2 \\
& \leq 5R(t-s)^{2\gamma} + 5R(t-s)^2 + 20R^5 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \\
& \quad + 5 \left(\int_s^t (t-u)^{2\delta} \mathbb{E} \left\| (-A)^\delta B(X_u) \right\|_{HS(U_0, H)}^2 du \right) \\
& \quad + 5R^2 \left(\int_s^t \left\| (-A)^{-\beta} \right\|_{L(H)}^2 \mathbb{E} \|X_u - X_s\|_{V_\beta}^2 du \right)
\end{aligned}$$

for every $s, t \in [0, T]$ with $s \leq t$ and every $K \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned}
\mathbb{E} \left\| X_t - X_s - \int_s^t B(X_s) dW_u^K \right\|_H^2 & \leq 5R(t-s)^{2\gamma} + 5R(t-s)^2 \\
& \quad + 20R^5 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} + 5R \left(\int_s^t (t-u)^{2\delta} du \right) \\
& \quad + 5R^4 \left(\int_s^t \mathbb{E} \|X_u - X_s\|_{V_\beta}^2 du \right)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E} \left\| X_t - X_s - \int_s^t B(X_s) dW_u^K \right\|_H^2 & \leq 10R^3(t-s)^{2\gamma} + 20R^5 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \\
& \quad + 5R(t-s)^{(1+2\delta)} + 5R^4 \left(\int_s^t \mathbb{E} \|X_u - X_s\|_{V_\beta}^2 du \right)
\end{aligned}$$

for every $s, t \in [0, T]$ with $s \leq t$ and every $K \in \mathbb{N}$. This shows

$$\begin{aligned}
\mathbb{E} \left\| X_t - X_s - \int_s^t B(X_s) dW_u^K \right\|_H^2 & \leq 15R^3(t-s)^{2\gamma} + 20R^5 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \\
& \quad + 5R^4 \left(\int_s^t \left(\mathbb{E} \|X_u - X_s\|_{V_\beta}^4 \right)^{\frac{1}{2}} du \right)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E} \left\| X_t - X_s - \int_s^t B(X_s) dW_u^K \right\|_H^2 & \leq 15R^3(t-s)^{2\gamma} + 20R^5 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \\
& \quad + 5R^4 \left(\int_s^t \left(R(u-s)^{\min(4(\gamma-\beta), 2)} \right)^{\frac{1}{2}} du \right)
\end{aligned}$$

and hence

$$\begin{aligned} \mathbb{E} \left\| X_t - X_s - \int_s^t B(X_s) dW_u^K \right\|_H^2 &\leq 15R^3 (t-s)^{2\gamma} + 20R^5 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \\ &\quad + 5R^5 \left(\int_s^t (u-s)^{\min(2(\gamma-\beta), 1)} du \right) \end{aligned}$$

for every $s, t \in [0, T]$ with $s \leq t$ and every $K \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned} \mathbb{E} \left\| X_t - X_s - \int_s^t B(X_s) dW_u^K \right\|_H^2 &\leq 15R^3 (t-s)^{2\gamma} + 20R^5 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} + 5R^5 (t-s)^{(1+\min(2(\gamma-\beta), 1))} \\ &\leq 15R^3 (t-s)^{2\gamma} + 20R^5 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} + 5R^6 (t-s)^{\min(4(\gamma-\beta), 2)} \end{aligned}$$

and finally

$$\begin{aligned} \mathbb{E} \left\| X_t - X_s - \int_s^t B(X_s) dW_u^K \right\|_H^2 &\leq 20R^8 (t-s)^{\min(4(\gamma-\beta), 2\gamma)} + 20R^5 \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \quad (79) \end{aligned}$$

for every $s, t \in [0, T]$ with $s \leq t$ and every $K \in \mathbb{N}$. Now we prove (76). To this end note that

$$\begin{aligned} &\mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} B'(X_{lh}) \left(X_s - X_{lh} - \int_{lh}^s B(X_{lh}) dW_u^K \right) dW_s^K \right\|_H^2 \\ &\leq \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \mathbb{E} \left\| B'(X_{lh}) \left(X_s - X_{lh} - \int_{lh}^s B(X_{lh}) dW_u^K \right) \right\|_{HS(U_0, H)}^2 ds \\ &\leq R^2 \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \mathbb{E} \left\| X_s - X_{lh} - \int_{lh}^s B(X_{lh}) dW_u^K \right\|_H^2 ds \right) \end{aligned}$$

holds for every $m \in \{0, 1, \dots, M\}$ and every $M, K \in \mathbb{N}$. Hence, (79) yields

$$\begin{aligned}
& \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} B'(X_{lh}) \left(X_s - X_{lh} - \int_{lh}^s B(X_{lh}) dW_u^K \right) dW_s^K \right\|_H^2 \\
& \leq 20R^{10} \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \left((s-lh)^{\min(4(\gamma-\beta), 2\gamma)} + \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \right) ds \right) \\
& \leq 20R^{10} \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} (s-lh)^{\min(4(\gamma-\beta), 2\gamma)} ds \right) + 20R^{10} T \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha}
\end{aligned}$$

and therefore

$$\begin{aligned}
& \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} B'(X_{lh}) \left(X_s - X_{lh} - \int_{lh}^s B(X_{lh}) dW_u^K \right) dW_s^K \right\|_H^2 \\
& \leq 20R^{10} M h^{(1+\min(4(\gamma-\beta), 2\gamma))} + 20R^{11} \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha} \\
& \leq 20R^{11} h^{\min(4(\gamma-\beta), 2\gamma)} + 20R^{11} \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha}
\end{aligned}$$

and finally

$$\begin{aligned}
& \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} B'(X_{lh}) \left(X_s - X_{lh} - \int_{lh}^s B(X_{lh}) dW_u^K \right) dW_s^K \right\|_H^2 \\
& \leq \frac{20R^{13}}{M^{\min(4(\gamma-\beta), 2\gamma)}} + 20R^{11} \left(\sup_{j \in \mathcal{J} \setminus \mathcal{J}_K} \mu_j \right)^{2\alpha}
\end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M, K \in \mathbb{N}$.

6.6 Lipschitz estimates: Proof of (77)

Before we estimate $\mathbb{E} \|Z_m^{N,M,K} - Y_m^{N,M,K}\|_H^2$ for $m \in \{0, 1, \dots, M\}$ and for $N, M, K \in \mathbb{N}$, we need some preparations. More precisely, we have

$$\begin{aligned} & \mathbb{E} \left\| B'(X_{lh}) \int_{lh}^s B(X_{lh}) dW_u^K - B'(Y_l^{N,M,K}) \int_{lh}^s B(Y_l^{N,M,K}) dW_u^K \right\|_{HS(U_0, H)}^2 \\ &= \mathbb{E} \left\| B'(X_{lh}) \left(\sum_{\substack{j \in \mathcal{J}_K \\ \mu_j \neq 0}} \int_{lh}^s B(X_{lh}) g_j d\langle g_j, W_u \rangle_U \right) \right. \\ & \quad \left. - B'(Y_l^{N,M,K}) \left(\sum_{\substack{j \in \mathcal{J}_K \\ \mu_j \neq 0}} \int_{lh}^s B(Y_l^{N,M,K}) g_j d\langle g_j, W_u \rangle_U \right) \right\|_{HS(U_0, H)}^2 \end{aligned}$$

and hence

$$\begin{aligned} & \mathbb{E} \left\| B'(X_{lh}) \int_{lh}^s B(X_{lh}) dW_u^K - B'(Y_l^{N,M,K}) \int_{lh}^s B(Y_l^{N,M,K}) dW_u^K \right\|_{HS(U_0, H)}^2 \\ &= \mathbb{E} \left\| B'(X_{lh}) \left(\sum_{\substack{j \in \mathcal{J}_K \\ \mu_j \neq 0}} B(X_{lh}) g_j \langle g_j, W_s - W_{lh} \rangle_U \right) \right. \\ & \quad \left. - B'(Y_l^{N,M,K}) \left(\sum_{\substack{j \in \mathcal{J}_K \\ \mu_j \neq 0}} B(Y_l^{N,M,K}) g_j \langle g_j, W_s - W_{lh} \rangle_U \right) \right\|_{HS(U_0, H)}^2 \end{aligned}$$

and

$$\begin{aligned} & \mathbb{E} \left\| B'(X_{lh}) \int_{lh}^s B(X_{lh}) dW_u^K - B'(Y_l^{N,M,K}) \int_{lh}^s B(Y_l^{N,M,K}) dW_u^K \right\|_{HS(U_0, H)}^2 \\ &= \mathbb{E} \left\| \sum_{\substack{j \in \mathcal{J}_K \\ \mu_j \neq 0}} \left\{ B'(X_{lh}) (B(X_{lh}) g_j) \right. \right. \\ & \quad \left. \left. - B'(Y_l^{N,M,K}) (B(Y_l^{N,M,K}) g_j) \right\} \langle g_j, W_s - W_{lh} \rangle_U \right\|_{HS(U_0, H)}^2 \end{aligned}$$

for every $s \in [lh, (l+1)h]$, $l \in \{0, 1, \dots, M-1\}$ and every $M, K \in \mathbb{N}$. Moreover, we have

$$\begin{aligned} & \mathbb{E} \left\| B'(X_{lh}) \int_{lh}^s B(X_{lh}) dW_u^K - B'(Y_l^{N,M,K}) \int_{lh}^s B(Y_l^{N,M,K}) dW_u^K \right\|_{HS(U_0, H)}^2 \\ &= \sum_{\substack{j \in \mathcal{J}_K \\ \mu_j \neq 0}} \mathbb{E} \left\| \left\{ B'(X_{lh}) (B(X_{lh}) g_j) \right. \right. \\ & \quad \left. \left. - B'(Y_l^{N,M,K}) (B(Y_l^{N,M,K}) g_j) \right\} \langle g_j, W_s - W_{lh} \rangle_U \right\|_{HS(U_0, H)}^2 \end{aligned}$$

and

$$\begin{aligned} & \mathbb{E} \left\| B'(X_{lh}) \int_{lh}^s B(X_{lh}) dW_u^K - B'(Y_l^{N,M,K}) \int_{lh}^s B(Y_l^{N,M,K}) dW_u^K \right\|_{HS(U_0, H)}^2 \\ &= \sum_{\substack{j \in \mathcal{J}_K \\ \mu_j \neq 0}} \mathbb{E} \left\| B'(X_{lh}) (B(X_{lh}) g_j) - B'(Y_l^{N,M,K}) (B(Y_l^{N,M,K}) g_j) \right\|_{HS(U_0, H)}^2 \\ & \quad \cdot \mathbb{E} |\langle g_j, W_s - W_{lh} \rangle_U|^2 \end{aligned}$$

for every $s \in [lh, (l+1)h]$, $l \in \{0, 1, \dots, M-1\}$ and every $M, K \in \mathbb{N}$. This shows

$$\begin{aligned} & \mathbb{E} \left\| B'(X_{lh}) \int_{lh}^s B(X_{lh}) dW_u^K - B'(Y_l^{N,M,K}) \int_{lh}^s B(Y_l^{N,M,K}) dW_u^K \right\|_{HS(U_0, H)}^2 \\ &= \sum_{\substack{j \in \mathcal{J}_K \\ \mu_j \neq 0}} \mu_j \cdot \mathbb{E} \left\| B'(X_{lh}) (B(X_{lh}) g_j) - B'(Y_l^{N,M,K}) (B(Y_l^{N,M,K}) g_j) \right\|_{HS(U_0, H)}^2 \\ & \quad \cdot (s - lh) \end{aligned}$$

and therefore

$$\begin{aligned} & \mathbb{E} \left\| B'(X_{lh}) \int_{lh}^s B(X_{lh}) dW_u^K - B'(Y_l^{N,M,K}) \int_{lh}^s B(Y_l^{N,M,K}) dW_u^K \right\|_{HS(U_0, H)}^2 \\ &\leq \sum_{\substack{j, k \in \mathcal{J} \\ \mu_j, \mu_k \neq 0}} \mu_j \mu_k \mathbb{E} \left\| B'(X_{lh}) (B(X_{lh}) g_j) g_k - B'(Y_l^{N,M,K}) (B(Y_l^{N,M,K}) g_j) g_k \right\|_H^2 \\ & \quad \cdot (s - lh) \end{aligned}$$

for every $s \in [lh, (l+1)h]$, $l \in \{0, 1, \dots, M-1\}$ and every $M, K \in \mathbb{N}$. Hence, we obtain

$$\begin{aligned} & \mathbb{E} \left\| B'(X_{lh}) \int_{lh}^s B(X_{lh}) dW_u^K - B'(Y_l^{N,M,K}) \int_{lh}^s B(Y_l^{N,M,K}) dW_u^K \right\|_{HS(U_0, H)}^2 \\ &= (s - lh) \cdot \mathbb{E} \left\| B'(X_{lh}) B(X_{lh}) - B'(Y_l^{N,M,K}) B(Y_l^{N,M,K}) \right\|_{HS^{(2)}(U_0, H)}^2 \end{aligned}$$

and finally

$$\begin{aligned} & \mathbb{E} \left\| B'(X_{lh}) \int_{lh}^s B(X_{lh}) dW_u^K - B'(Y_l^{N,M,K}) \int_{lh}^s B(Y_l^{N,M,K}) dW_u^K \right\|_{HS(U_0, H)}^2 \\ & \leq c^2 \cdot (s - lh) \cdot \mathbb{E} \left\| X_{lh} - Y_l^{N,M,K} \right\|_H^2 \leq R^3 \cdot \mathbb{E} \left\| X_{lh} - Y_l^{N,M,K} \right\|_H^2 \quad (80) \end{aligned}$$

for every $s \in [lh, (l+1)h]$, $l \in \{0, 1, \dots, M-1\}$ and every $M, K \in \mathbb{N}$. Additionally, (69) shows

$$\begin{aligned} & \mathbb{E} \left\| Z_m^{N,M,K} - Y_m^{N,M,K} \right\|_H^2 \\ & \leq 3 \cdot \mathbb{E} \left\| P_N \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} \left(F(X_{lh}) - F(Y_l^{N,M,K}) \right) ds \right) \right\|_H^2 \\ & \quad + 3 \cdot \mathbb{E} \left\| P_N \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} \left(B(X_{lh}) - B(Y_l^{N,M,K}) \right) dW_s^K \right) \right\|_H^2 \\ & \quad + 3 \cdot \mathbb{E} \left\| P_N \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} \left(B'(X_{lh}) \int_{lh}^s B(X_{lh}) dW_u^K \right. \right. \right. \\ & \quad \left. \left. \left. - B'(Y_l^{N,M,K}) \int_{lh}^s B(Y_l^{N,M,K}) dW_u^K \right) dW_s^K \right) \right\|_H^2 \end{aligned}$$

and hence

$$\begin{aligned}
& \mathbb{E} \left\| Z_m^{N,M,K} - Y_m^{N,M,K} \right\|_H^2 \\
& \leq 3 \cdot \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} \left(F(X_{lh}) - F(Y_l^{N,M,K}) \right) ds \right\|_H^2 \\
& \quad + 3 \cdot \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} \left(B(X_{lh}) - B(Y_l^{N,M,K}) \right) dW_s^K \right\|_H^2 \\
& \quad + 3 \cdot \mathbb{E} \left\| \sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} e^{A(m-l)h} \left(B'(X_{lh}) \int_{lh}^s B(X_{lh}) dW_u^K \right. \right. \\
& \quad \quad \left. \left. - B'(Y_l^{N,M,K}) \int_{lh}^s B(Y_l^{N,M,K}) dW_u^K \right) dW_s^K \right\|_H^2
\end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $N, M, K \in \mathbb{N}$. Moreover, we have

$$\begin{aligned}
& \mathbb{E} \left\| Z_m^{N,M,K} - Y_m^{N,M,K} \right\|_H^2 \\
& \leq 3Mh^2 \left(\sum_{l=0}^{m-1} \mathbb{E} \left\| e^{A(m-l)h} \left(F(X_{lh}) - F(Y_l^{N,M,K}) \right) \right\|_H^2 \right) \\
& \quad + 3 \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \mathbb{E} \left\| e^{A(m-l)h} \left(B(X_{lh}) - B(Y_l^{N,M,K}) \right) \right\|_{HS(U_0, H)}^2 ds \right) \\
& \quad + 3 \left(\sum_{l=0}^{m-1} \int_{lh}^{(l+1)h} \mathbb{E} \left\| B'(X_{lh}) \int_{lh}^s B(X_{lh}) dW_u^K \right. \right. \\
& \quad \quad \left. \left. - B'(Y_l^{N,M,K}) \int_{lh}^s B(Y_l^{N,M,K}) dW_u^K \right\|_{HS(U_0, H)}^2 ds \right)
\end{aligned}$$

and due to (80) we obtain

$$\begin{aligned}
& \mathbb{E} \left\| Z_m^{N,M,K} - Y_m^{N,M,K} \right\|_H^2 \leq 3Th \left(\sum_{l=0}^{m-1} \mathbb{E} \left\| F(X_{lh}) - F(Y_l^{N,M,K}) \right\|_H^2 \right) \\
& \quad + 3h \left(\sum_{l=0}^{m-1} \mathbb{E} \left\| B(X_{lh}) - B(Y_l^{N,M,K}) \right\|_{HS(U_0, H)}^2 \right) \\
& \quad + 3R^3h \left(\sum_{l=0}^{m-1} \mathbb{E} \left\| X_{lh} - Y_l^{N,M,K} \right\|_H^2 \right)
\end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $N, M, K \in \mathbb{N}$. Finally, we obtain

$$\begin{aligned} \mathbb{E} \|Z_m^{N,M,K} - Y_m^{N,M,K}\|_H^2 &\leq 9R^3 h \left(\sum_{l=0}^{m-1} \mathbb{E} \|X_{lh} - Y_l^{N,M,K}\|_H^2 \right) \\ &\leq \frac{9R^4}{M} \left(\sum_{l=0}^{m-1} \mathbb{E} \|X_{lh} - Y_l^{N,M,K}\|_H^2 \right) \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $N, M, K \in \mathbb{N}$.

6.7 Iterated integral identity: Proof of (65)

First of all, we have

$$\begin{aligned} &\int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} B'(Y_m^{N,M,K}) \left(\int_{\frac{mT}{M}}^s B(Y_m^{N,M,K}) dW_u^K \right) dW_s^K \\ &= \sum_{\substack{j,k \in \mathcal{J}_K \\ \mu_j, \mu_k \neq 0}} \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} B'(Y_m^{N,M,K}) \left(\int_{\frac{mT}{M}}^s B(Y_m^{N,M,K}) g_k d\langle g_k, W_u \rangle_U \right) g_j d\langle g_j, W_s \rangle_U \\ &= \sum_{\substack{j,k \in \mathcal{J}_K \\ \mu_j, \mu_k \neq 0}} B'(Y_m^{N,M,K}) \left(B(Y_m^{N,M,K}) g_k \right) g_j \cdot \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} \int_{\frac{mT}{M}}^s d\langle g_k, W_u \rangle_U d\langle g_j, W_s \rangle_U \end{aligned}$$

\mathbb{P} -a.s. and hence

$$\begin{aligned} &\int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} B'(Y_m^{N,M,K}) \left(\int_{\frac{mT}{M}}^s B(Y_m^{N,M,K}) dW_u^K \right) dW_s^K \\ &= \sum_{\substack{j \in \mathcal{J}_K \\ \mu_j \neq 0}} B'(Y_m^{N,M,K}) \left(B(Y_m^{N,M,K}) g_j \right) g_j \cdot \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} \int_{\frac{mT}{M}}^s d\langle g_j, W_u \rangle_U d\langle g_j, W_s \rangle_U \\ &+ \sum_{\substack{j,k \in \mathcal{J}_K \\ \mu_j, \mu_k \neq 0 \\ j \neq k}} B'(Y_m^{N,M,K}) \left(B(Y_m^{N,M,K}) g_k \right) g_j \cdot \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} \int_{\frac{mT}{M}}^s d\langle g_k, W_u \rangle_U d\langle g_j, W_s \rangle_U \end{aligned}$$

\mathbb{P} -a.s. for all $m \in \{0, 1, \dots, M-1\}$ and all $N, M, K \in \mathbb{N}$. Moreover, since the bilinear operator $B'(Y_m^{N,M,K}) B(Y_m^{N,M,K}) \in HS^{(2)}(U_0, H)$ is symmetric (see Assumption 3) and since

$$\int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} \int_{\frac{mT}{M}}^s d\langle g_j, W_u \rangle_U d\langle g_j, W_s \rangle_U = \frac{1}{2} \left((\langle g_j, \Delta W_m^{M,K} \rangle_U)^2 - \frac{T\mu_j}{M} \right)$$

\mathbb{P} -a.s. holds for all $j \in \mathcal{J}_K$, $m \in \{0, 1, \dots, M-1\}$ and all $N, M, K \in \mathbb{N}$ (see (3.6) in Section 10.3 in [36]), we obtain

$$\begin{aligned}
& \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} B'(Y_m^{N,M,K}) \left(\int_{\frac{mT}{M}}^s B(Y_m^{N,M,K}) dW_u^K \right) dW_s^K \\
&= \frac{1}{2} \sum_{\substack{j \in \mathcal{J}_K \\ \mu_j \neq 0}} B'(Y_m^{N,M,K}) \left(B(Y_m^{N,M,K}) g_j \right) g_j \left((\langle g_j, \Delta W_m^{M,K} \rangle_U)^2 - \frac{T\mu_j}{M} \right) \\
&+ \frac{1}{2} \sum_{\substack{j,k \in \mathcal{J}_K \\ \mu_j, \mu_k \neq 0 \\ j \neq k}} B'(Y_m^{N,M,K}) \left(B(Y_m^{N,M,K}) g_k \right) g_j \\
&\cdot \left(\int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} \int_{\frac{mT}{M}}^s d\langle g_k, W_u \rangle_U d\langle g_j, W_s \rangle_U + \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} \int_{\frac{mT}{M}}^s d\langle g_j, W_u \rangle_U d\langle g_k, W_s \rangle_U \right)
\end{aligned}$$

\mathbb{P} -a.s. for all $m \in \{0, 1, \dots, M-1\}$ and all $N, M, K \in \mathbb{N}$. The fact

$$\begin{aligned}
& \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} \int_{\frac{mT}{M}}^s d\langle g_k, W_u \rangle_U d\langle g_j, W_s \rangle_U + \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} \int_{\frac{mT}{M}}^s d\langle g_j, W_u \rangle_U d\langle g_k, W_s \rangle_U \\
&= \langle g_k, \Delta W_m^{M,K} \rangle_U \langle g_j, \Delta W_m^{M,K} \rangle_U \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

for all $j \in \mathcal{J}_K$, $m \in \{0, 1, \dots, M-1\}$ and all $N, M, K \in \mathbb{N}$ (see (3.15) in Section 10.3 in [36]) then yields

$$\begin{aligned}
& \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} B'(Y_m^{N,M,K}) \left(\int_{\frac{mT}{M}}^s B(Y_m^{N,M,K}) dW_u^K \right) dW_s^K \\
&= \frac{1}{2} \sum_{\substack{j,k \in \mathcal{J}_K \\ \mu_j, \mu_k \neq 0}} B'(Y_m^{N,M,K}) \left(B(Y_m^{N,M,K}) g_k \right) g_j \langle g_k, \Delta W_m^{M,K} \rangle_U \langle g_j, \Delta W_m^{M,K} \rangle_U \\
&\quad - \frac{T}{2M} \sum_{\substack{j \in \mathcal{J}_K \\ \mu_j \neq 0}} \mu_j B'(Y_m^{N,M,K}) \left(B(Y_m^{N,M,K}) g_j \right) g_j \\
&= \frac{1}{2} B'(Y_m^{N,M,K}) \left(B(Y_m^{N,M,K}) \Delta W_m^{M,K} \right) \Delta W_m^{M,K} \\
&\quad - \frac{T}{2M} \sum_{\substack{j \in \mathcal{J}_K \\ \mu_j \neq 0}} \mu_j B'(Y_m^{N,M,K}) \left(B(Y_m^{N,M,K}) g_j \right) g_j \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

for all $m \in \{0, 1, \dots, M-1\}$ and all $N, M, K \in \mathbb{N}$ which shows (65).

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